Equivariant maps between $C_{2p}$-representation spheres for an odd prime $p$

Ikumitsu NAGASAKI 1)

Abstract. In the previous research, we discussed a necessary and sufficient condition for the existence of a $G$-map between unitary representation spheres of a cyclic group $C_{pq}$, where $p$ and $q$ are distinct primes. In this paper, we study a similar problem for orthogonal representation spheres of $C_{pq}$. In particular, we treat the case of $C_{2p}$, where $p$ is an odd prime. As a result, we show that some results in the unitary case do not hold in the orthogonal case.

1. Background

Let $G$ be a finite group. In equivariant topology, the following question is fundamental and important for application to topological problems.

Question. Given $G$-representation spheres $S(V)$ and $S(W)$, does there exist a $G$-map $f : S(V) \to S(W)$ or not?

For example, a kind of non-existence result on an equivariant map plays a crucial role in the proof of Furuta’s 10/8-theorem [3]. If the $G$-fixed point set $S(W)^G$ is not empty, then clearly a $G$-map always exists, and if $S(V)^G \neq \emptyset$ and $S(W)^G = \emptyset$, then there are no $G$-maps. Therefore we assume that representations $V$ and $W$ are $G$-fixed-point-free; i.e., $V^G = W^G = 0$ unless otherwise stated.

Equivariant obstruction theory provides several results on the above question, for example, see [5, 6, 9]. However a complete answer is not obtained at present since the computation of obstruction classes is difficult in general. In [7], we have treated unitary representations of a cyclic group $C_{pq}$, where $p$, $q$ are distinct primes, and give an answer of the question as follows:

**Proposition 1.1 ([7]).** Let $G = C_{pq}$ and $V$ and $W$ be unitary $G$-representations with $V^G = W^G = 0$. Then there exists a $G$-map $f : S(V) \to S(W)$ if and only if the following conditions hold.

1) Department of Mathematics, Kyoto Prefectural University of Medicine, 1-5 Shimogamohangicho, Sakyo-ku, Kyoto 606-0823, Japan. e-mail: nagasaki@koto.kpu-m.ac.jp

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(1) \( \dim V^C \leq \dim W^C \) and \( \dim V^C_\mu \leq \dim W^C_\mu \).

(2) If \( \dim W^C = 0 \) or \( \dim W^C_\mu = 0 \), then \( \dim V \leq \dim W \).

In this paper, we consider orthogonal \( C_{pq} \)-representations. If \( p, q \) are odd primes, then the same result holds because any orthogonal representation of a finite group of odd order has a complex structure. On the other hand, if \( q = 2 \) and \( p \) is an odd prime, then Proposition 1.1 does not hold as mentioned in section 3. One of the purposes of this paper is to present such a counterexample.

2. Preliminary facts on \( C_n \)-maps

Let us recall irreducible orthogonal representations of a cyclic group \( C_n \) of order \( n \). Let \( a \) be a generator of \( C_n \). By Serre [8], irreducible unitary \( C_n \)-representations \( U_k (= \mathbb{C}) \), \( k \in \mathbb{Z}/n \), are given by \( az = \xi_n^k z \), \( z \in U_k \), where \( \xi_n = \exp(2\pi \sqrt{-1}/n) \). We set \( T_k = \text{res}_U U_k \). Then (2-dimensional) irreducible orthogonal \( C_n \)-representations are given by \( T_k \) when \( k \not\equiv 0 \pmod{n} \) or \( k \not\equiv n/2 \pmod{n} \) if \( n \) is even. If \( n \) is even and \( k = n/2 \), there is a 1-dimensional representation \( \mathbb{R} \), where \( \varepsilon : C_n \to \{\pm 1\} \) is the sign homomorphism. The action of \( C_n \) on \( \mathbb{R} \) is given by \( gx = \varepsilon(g)x \) for any \( g \in C_n \) and \( x \in \mathbb{R} \); in particular, \( ax = -x \). Not that \( T_k \cong T_l \) as orthogonal representations if \( kl \equiv 1 \pmod{n} \). Note \( T_0 = 2\mathbb{R} \) and \( T_{n/2} = 2\mathbb{R} \) if \( n \) is even. Summarizing these facts, we have

**Proposition 2.1.** All orthogonal irreducible representations of \( C_n \) are given as follows.

1. When \( n \) is odd, there are \( (n-1)/2 \) 2-dimensional irreducible representations \( T_k \) \( (1 \leq k \leq (n-1)/2) \) and there is a 1-dimensional trivial representation \( \mathbb{R} \).
2. When \( n \) is even, there are \( (n-2)/2 \) 2-dimensional irreducible representations \( T_k \) \( (1 \leq k \leq n/2-1) \) and there are 1-dimensional representations \( \mathbb{R} \) and \( \mathbb{R} \).

We next discuss the existence of a \( G \)-map between \( S(U_k) \) and \( S(U_l) \), where \( k, l \not\equiv 0 \pmod{n} \). Some special cases are described in [7].

**Proposition 2.2.** Let \( G = C_n \). There exists a \( G \)-map \( f : S(U_k) \to S(U_l) \) if and only if \( (k,n) \) divides \( (l,n) \), where \( (k,n) \) denotes the greatest common divisor of \( k \) and \( n \).

**Proof.** Set \( d = (k,n) \) and \( e = (l,n) \). If there exists a \( G \)-map \( f : S(U_k) \to S(U_l) \), then for any \( x \in S(U_k) \), it follows that

\[
G_x = \text{Ker} U_k = \langle a^{n/d} \rangle \cong C_d \leq G_f(x) = \text{Ker} U_l = \langle a^{n/e} \rangle \cong C_e.
\]
Hence \( d \) divides \( e \).

We show the converse. Assume that \( d = (k, l) \) divides \( e = (l, n) \). Recall that the action of the generator \( a \in G \) on \( U_k \) is given by \( az = \xi_k^h z, z \in U_k \). Therefore one see that \( K := \text{Ker} U_k = C_d \leq G \). Furthermore \( U_k \) is regarded as \( G/K \)-representation and \( G/K \cong C_{n/d} \) acts freely on \( U_k \). By assumption, it follows that \( L := \text{Ker} U_l \geq K \) and \( G/K \) acts on \( U_l \) with the kernel \( L/K \). If there exists a \( G/K \)-map, then one may assume that \((k, n) = 1 \). Take an integer \( k' \) with \( kk' \equiv 1 \mod n \) and define a map \( f : S(U_k) \to S(U_l) \) by \( f(z) = z^{k'l} \) for \( z \in U_k \). One can easily see that this map is \( G \)-equivariant. \( \square \)

**Remark.** If we set \((0, n) = n \), then the above proposition holds for \( k = 0 \) or \( l = 0 \).

By restricting the ground field \( \mathbb{C} \) to \( \mathbb{R} \), we obtain

**Corollary 2.3.** There exists a \( G \)-map \( f : S(T_k) \to S(T_l) \) if and only if \((k, n) \) divides \((l, n) \).

### 3. The case of \( C_{2p} \)

In this section, \( G \) is a cyclic group \( C_{2p} \) of order \( 2p \), where \( p \) is an odd prime. Let \( V \) and \( W \) be orthogonal \( G \)-representations with \( V^G = W^G = 0 \). We consider the question whether a \( G \)-map from \( S(V) \) to \( S(W) \) exists. The non-trivial irreducible representations are:

\[
T_i \quad (1 \leq i \leq p - 1), \quad \mathbb{R}_\varepsilon.
\]

Note that \( \text{Ker} T_i = 1 \) for odd \( i \) and \( \text{Ker} T_i = C_2 \) for even \( i \). By the same argument of [7], we obtain the following fact.

**Proposition 3.1.** If there exists a \( G \)-map \( f : S(V) \to S(W) \), then

\[(C1) \quad \dim V^{C_p} \leq \dim W^{C_p} \quad \text{and} \quad \dim V^{C_2} \leq \dim W^{C_2}.
\]

\[(C2) \quad \text{If } \dim W^{C_p} = 0 \quad \text{or} \quad \dim W^{C_2} = 0, \quad \text{then } \dim V \leq \dim W.
\]

We would like to consider the converse. We first note that

**Theorem 3.2.** In addition to (C1) and (C2), if \( \dim W^{C_p} \geq 2 \) is satisfied, then there exists a \( G \)-map \( f : S(V) \to S(W) \).

**Proof.** Since the proof is similar with [7], we only give an outline. Let

\[
V = a_1 T_1 \oplus \cdots \oplus a_{p-1} T_{p-1} \oplus c \mathbb{R}_\varepsilon \quad (a_i \geq 0, \ c \geq 0)
\]
be the irreducible decomposition of $V$. For any subgroup $K$ of $G$, let $V(K)$ denote the direct sum of irreducible representations with kernel $K$ in the irreducible decomposition of $V$. So we have

$$V(1) = \bigoplus_{i: \text{odd}} a_i T_i \quad V(C_2) = \bigoplus_{i: \text{even}} a_i U_i \quad V(C_p) = c \mathbb{R}.$$ 

Hence $V = V(1) \oplus V(C_2) \oplus V(C_p)$. Similarly we obtain $W = W(1) \oplus W(C_2) \oplus W(C_p)$. Note also that $V^{C_2} = V(C_2)$ and $V^{C_p} = V(C_p)$. If dim $W(C_2) = 0$, then $V(C_2) = 0$ and one can easily see the existence of a $G$-map from condition (C2). Assume dim $W(C_2) > 0$; in fact dim $W(C_2) \geq 2$. Let $C$ be $C_p$ or $C_2$. Since $W^C = W(C)$, it follows that dim $W^C \geq 2$ by assumption. Then one can find a $G$-map $h : S(W) \to S(W)$ with deg $h = 0$ using equivariant obstruction theory.

One can construct a $G$-map $f^{>1} : S(V)^{>1} \to S(W)$, where

$$S(V)^{>1} = S(V)^{C_p} \coprod_{i: \text{even}} S(V)^{C_2}$$

is the singular set of $S(V)$. By composing $f^{>1}$ with $h$, it follows that the equivariant obstruction to the extension of $h \circ f^{>1}$ vanishes. Hence there exists a $G$-map $f : S(V) \to S(W)$.

From the above proposition, the remaining case is

$$\dim W^{C_p} = 1 \text{ and } \dim W^{C_2} > 0.$$ 

**Proposition 3.3.** In this case, for any $G$-map $h : S(W) \to S(W)$, it follows that $\deg h^{C_p} = \pm 1$ and $\deg h \equiv \pm 1 \mod p$. In particular $\deg h \neq 0$.

**Proof.** Note that $\deg h^G = 1$ since $S(W)^G = \emptyset$. Since

$$h^{C_p} : S(W)^{C_p} = S^0 \to S(W)^{C_p} = S^0$$

is $G/C_p \cong C_2$-map and $C_2$ acts freely on $S(W)^{C_p}$, it follows that $\deg h^{C_p} = \pm 1$. By the Burnside relation, see [1, 2], we obtain $\deg h^{C_2} \equiv 1 \mod p$ and

$$\deg h + \deg h^{C_2} + (p-1) \deg h^{C_p} + (p-1) \deg h^G \equiv 0 \mod 2p.$$ 

Reducing this to mod $p$, we have

$$\deg h \equiv - \deg h^{C_2} + \deg h^{C_p} + \deg h^G \mod p \equiv \deg h^{C_p} \equiv \pm 1 \mod p.$$ 

□
This result means that the argument in Theorem 3.2 is not available; namely, we cannot prove the vanishing of the obstruction.

As an easy corollary, we obtain a variation of Borsuk-Ulam results, cf. [10].

**Corollary 3.4.** In the above situation, if $W \subseteq U$, then there are no $G$-maps from $S(U)$ to $S(W)$.

**Proof.** If there exists a $G$-map $f : S(U) \to S(W)$, then $h := i \circ f : S(W) \to S(W)$ has a non-zero degree, where $i$ is the inclusion. On the other hand, by dimensional reason, $\deg h = 0$; this is a contradiction. □

Under the condition $\dim V^{C_2} \leq \dim W^{C_2}$, we next discuss the question in the following two cases:

1. $\dim V^{C_p} = \dim W^{C_p} = 1$.
2. $\dim V^{C_p} = 0$ and $\dim W^{C_p} = 1$.

we here provide two examples. The first example shows that Proposition 1.1 does not hold in orthogonal case. Let $G = C_{2p}$ as before.

**Example 3.5.** Let $V = T_1 \oplus T_1 \oplus \mathbb{R}_e$ and $W = T_2 \oplus \mathbb{R}_e$. There are no $G$-maps from $S(V)$ to $S(W)$.

**Proof.** Suppose that there exists a $G$-map $f : S(V) \to S(W)$. Set $U = T_1 \oplus \mathbb{R}_e$. Consider a $C_p$-map $h := \text{res}_{C_p} f|_{S(U)} : S(U) \to S(W)$. Then $\deg h \equiv \pm 2 \mod p$. In fact, $h^{C_p} = \pm id$ and there is a $G$-map $k : S(U) \to S(W)$ with $\deg k = \pm 2$ and $k^{C_p} = h^{C_p}$.

By a result of equivariant obstruction theory [1, 2], for any $G$-map $h : S(U) \to S(W)$, it follows that $\deg h - \deg k \equiv 0 \mod p$. As a result, we have $\deg h \neq 0$.

On the other hand, there exists a 3-disk $D^3 \subset S(V)$ such that $\partial D^3 = S(U)$, and so $\deg h = 0$; this is a contradiction. In fact, $D^3$ can be taken as follows. Note that

$$S(V) \cong S(T_1) \times D(U) \cup D(T_1) \times S(U) \subset \mathbb{R}^2 \times \mathbb{R}^3$$

$$= \{(x, y) \mid \|x\| = 1, \|y\| \leq 1\} \cup \{(x, y) \mid \|x\| \leq 1, \|y\| = 1\},$$
and $S(U)$ is regarded as $0 \times S(U) = \{(0, y) \mid \|y\| = 1\}$. Consider the following sets:

$$D = \{(1, y) \mid \|y\| \leq 1\} \cong D^3$$
and $C = \{(t, y), 0 \leq t \leq 1, \|y\| = 1\} \cong I \times S(U)$.

By attaching $C$ to $D$ on $\partial D = 1 \times S(U)$, we obtain $X = C \cup_{1 \times S(U)} D$. Then $X \cong D^3$ and $\partial X = S(U)$. □
Remark. By [4], $h$ is equivariantly desuspended to a map $h' : S(T_1) \to S(T_2)$ as $C_p$-maps. This fact also shows that $\deg h \neq 0$.

The next example is a counterexample to the Borsuk-Ulam theorem.

**Example 3.6.** Let $V = T_1 \oplus T_2$ and $W = T_2 \oplus \mathbb{R}_\varepsilon$. There exists a $G$-map $f : S(V) \to S(W)$.

**Proof.** Note that

$$S(V) \cong S(T_1) \ast S(T_2) = D(T_1) \times S(T_2) \cup S(T_1) \times D(T_2).$$

Take a $G$-map

$$\bar{h} : S(T_2) = 0 \times S(T_2) (\subset S(V)) \to S(T_2) = S(T_2) \times 0 (\subset S(W))$$

with even degree. For example, one can take a $G$-map defined by $\bar{h}(z) = z^{p+1}$ whose degree $\deg \bar{h} = p + 1$ is even.

In the same way as the proof of Example 3.5, we can take a 2-disk $D^2 \subset S(V)$ whose boundary is $S(T_2) = 0 \times S(T_2) \cong S^1$. Set $S_2 := D^2 \cup aD^2 \subset S(V)$ which is homeomorphic to a 2-sphere $S^2$, where $a$ is a generator of $G$.

Since $\bar{h}$ is of even degree, we can construct a (non-equivariant) map $h : S_2 \to S(W)$ extending $\bar{h}$ such that $\deg h = 0$. In fact, we can change the degree of $h$ by any even number.

Finally, take a 3-disk $D^3 \subset S(V)$ which is a region between $D^2$ and $aD^2$; hence $S_2$ is the boundary of $D^3$. Then it follows that

$$S(V) = \bigcup_{g \in G} (gD^3).$$

Since $\deg h = 0$, there exists a (non-equivariant) map $\tilde{h} : D^3 \to S(W)$ extending $h$. Using this, we can define a $G$-map

$$f := \bigcup_{g \in G} (g \cdot \tilde{h}) : S(V) = \bigcup_{g \in G} (gD^3) \to S(W)$$

by $f(x) = g\tilde{h}(g^{-1}x)$ for any $x \in gD^3$. One can easily check that this map is a well-defined $G$-map. □

**Corollary 3.7.** There exists a $G$-map $g : S(T_1 \oplus T_1) \to T_2 \oplus \mathbb{R}_\varepsilon$.

**Proof.** By Proposition 2.2, there is a $G$-map $k : S(T_1) \to S(T_2)$. This $G$-map $k$ induces a $G$-map $h = id \ast k : S(T_1 \oplus T_1) \to S(T_1 \oplus T_2)$. Composing $h$ with $f$, we obtain a $G$-map $g = f \circ h$. □
Remark. There are no \( G \)-maps from \( S(T_1 \oplus T_1) \) to \( S(T_1 \oplus \mathbb{R}_e) \) because the condition (C2) of Proposition 3.1 is not satisfied.

Conflict of Interest. The author has no conflicts of interest directly relevant to the content of this article.

References


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