The equivariant level and colevel of representation spheres

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Abstract. We introduce the equivariant level $l_G(X)$ and colevel $cl_G(X)$ of a G-space X. These are generalizations of classical invariants for spaces with free involutions. We first provide general properties of $l_G(X)$ and $cl_G(X)$. Secondly we provide some computations or estimates of $l_G(X)$ and $cl_G(X)$ when G is a finite cyclic group C_{pq} of order pq, where p and q are primes and X is a G-representation sphere.

1. Introduction

Let G be a compact Lie group and X a (non-empty) G-space X. Let V and W be (finite dimensional) fixed-point-free orthogonal representations. Let $L_G(X)$ be the set of G-representations W such that there exists a G-map $f : X \to S(W)$, and $CL_G(X)$ the set of G-representations V such that there exists a G-map $g : S(V) \to X$. We define the equivariant level $l_G(X)$ and colevel $cl_G(X)$ of X as follows.

Definition.

(1) G-level: $l_G(X) := \inf\{\dim W \mid W \in L_G(X)\}$.

(2) G-colevel: $cl_G(X) := \sup\{\dim V \mid V \in CL_G(X)\}$.

If $L_G(X) = \emptyset$, e.g., $X^G \neq \emptyset$, then we set $l_G(X) = \infty$. The G-level $l_G(X)$ cannot be 0 since if $W = 0$, then there are no G-maps to $S(W) = \emptyset$. Therefore $1 \leq l_G(X) \leq \infty$.

If $V = 0$, then $S(V) = \emptyset$. We regard $\emptyset : \emptyset \to X$ as a G-map. Hence $\emptyset \in CL_G(X)$ and we see $0 \leq cl_G(X) \leq \infty$. Also if $X^G \neq \emptyset$, then there is a G-map $f : S(V) \to X$ for any V, and hence $cl_G(X) = \infty$.

The G-level $l_G(X)$ is a generalization of the level in [7] or the coindex in [4], [5] for spaces with free involutions. The G-colevel $cl_G(X)$ is a generalization of the index in [4], [5] for spaces with free involutions.

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As related invariants, there are the genera $\gamma_G(X)$ and $\tilde{\gamma}_G(X)$ of X defined by [1]. We mention a relation between $l_G(X)$ and $\tilde{\gamma}_G(X)$ or $\gamma_G(X)$ in the next section.

One of purposes of this paper is to provide general properties of $l_G(X)$ and $cl_G(X)$. Secondly we provide some computations or estimates when G is a finite cyclic group of order pq, where p and q are primes.

2. **General properties of the equivariant level and colevel**

Since $l_G(X) = cl_G(X) = \infty$ when $X \neq \emptyset$, we hereafter assume that X is a fixed-point-free G-space, i.e., $X^G = \emptyset$. We begin by the following results.

Proposition 2.1. *Let X and Y be G-spaces and $f : X \to Y$ a G-map.*

(1) $l_G(X) \leq l_G(Y)$.
(2) $cl_G(X) \leq cl_G(Y)$.

Proof. (1) Let $l_G(Y) = k$. There exists a G-map $g : Y \to S(W)$, $\dim W = k$, realizing the level k. Composing g with f, one obtains a G-map $g \circ f : X \to S(W)$. This implies $l_G(X) \leq l_G(Y)$.

(2) Let $cl_G(X) = k$. There exists a G-map $g : S(V) \to X$, $\dim V = k$, realizing the colevel k and then one obtains a G-map $f \circ g : S(V) \to Y$. This implies $cl_G(X) \leq cl_G(Y)$. □

Proposition 2.2. *Let H be a closed normal subgroup of G and $\pi : G \to Q = G/H$ the projection. Let X be a Q-space and $\mathrm{Inf}_Q^G X$ is the inflation via π.*

(1) $l_G(\mathrm{Inf}_Q^G X) = l_Q(X)$.
(2) $cl_G(\mathrm{Inf}_Q^G X) \geq cl_Q(X)$.

Proof. (1) Let $l_Q(X) = k$ and $f : X \to S(W)$, $\dim W = k$, be a Q-map realizing the level k. Then $\mathrm{Inf}_Q^G f : \mathrm{Inf}_Q^G X \to S(\mathrm{Inf}_Q^G W)$ is a G-map. This implies $l_G(\mathrm{Inf}_Q^G X) \leq l_Q(X)$. Next let $l(\mathrm{Inf}_Q^G X) = k$ and $f : \mathrm{Inf}_Q^G X \to S(W)$ be a G-map realizing k. Since $G_x \geq H$ for $x \in \mathrm{Inf}_Q^G X$, it follows that $f(\mathrm{Inf}_Q^G X) \subset S(W)^H$. Since W^H is a G-representation, by the minimality of k, one sees that $W^H = W$. Therefore one obtains a Q-map $f^H : X \to S(W)$ and thus $l_Q(X) \leq k = l_G(\mathrm{Inf}_Q^G X)$. Thus (1) holds.

(2) Let $cl_Q(X) = k$ and $f : S(V) \to X$, $\dim V = k$, be a Q-map realizing the colevel k. Then $\mathrm{Inf}_Q^G f : S(\mathrm{Inf}_Q^G V) \to \mathrm{Inf}_Q^G X$ is a G-map. This implies $cl_G(\mathrm{Inf}_Q^G X) \geq cl_Q(X)$. □

Proposition 2.3. *Let H be a closed subgroup of G and X a G-space.*

(1) *If H is normal and $X^H = \emptyset$, then $l_H(\mathrm{Res}_H X) \leq l_G(X)$.*
(2) *If $X^H = \emptyset$, then $cl_H(\mathrm{Res}_H X) \geq cl_G(X)$.*

Proof. (1) Assume $l_G(X) = k$. Let $f : X \to S(W)$, $k = \dim W$, be a G-map realizing the level k. Then $\mathrm{Res}_H f : \mathrm{Res}_H X \to \mathrm{Res}_H S(W)$ is an H-map with $(\mathrm{Res}_H X)^H = \emptyset$. Since H is normal, f maps X into $S(W) \setminus S(W^H)$. Since $S(W) \setminus S(W^H)$ is H-homotopy equivalent to $S(W_H)$, where W_H is the orthogonal complement of V^H in V. Thus there exists an H-map $f' : \mathrm{Res}_H X \to S(V_H)$ with $S(V_H)^H = \emptyset$. Since $\dim V_H \leq k$, it follows that $l_H(\mathrm{Res}_H X) \leq l_G(X)$.

(2) Assume $cl_G(X) = k$. Let $f : S(V) \to X$, $k = \dim V$, be a G-map realizing the colevel k. Then $\mathrm{Res}_H f : S(\mathrm{Res}_H V) \to \mathrm{Res}_H X$ is an H-map. This implies $cl_H(\mathrm{Res}_H X) \geq cl_G(X)$. □

Proposition 2.4. *If X is a finite dimensional G-CW complex with finite orbit types, then $l_G(X) < \infty$.*

Proof. There exists a fixed-point-free representation W such that $\dim X^H \leq \dim S(W)^H$ for every $H \in \mathrm{Iso}(X)$. Then we see

$$H^k(X^H/WH, X^{>H}/WH; \pi_{k-1}(S(W))) = 0$$

for $1 \leq k \leq \dim X^H/WH$. By equivariant obstruction theory [6], one can construct a G-map $f : X \to S(W)$. Thus $l_G(X) \leq \dim W$. □

When X has infinitely many orbit types, $l_G(X)$ can be ∞. For example, let $G = S^1$ be a circle group and $H_k = C_k$ a finite cyclic subgroup of order $k \geq 1$. Set $X = \coprod_{q:\ \mathrm{prime}} G/H_q$, which is 1-dimensional G-CW complex with infinitely many orbit types. Since a representation sphere $S(W)$ has finitely many orbit types, see [3], and H_q are maximal isotropy subgroups in S^1, there are no G-maps $f : X \to S(W)$. This means $L_G(X) = \emptyset$ and $l_G(X) = \infty$ by definition.

We next provide some results obtaining from Borsuk-Ulam type theorems. Let G be an elementary abelian group C_p^k of rank k or a k-dimensional torus T^k. As is well-known, the Borsuk-Ulam theorem holds for these G, i.e., if there exists a G-map $f : S(V) \to S(W)$ between fixed-point-free representation spheres, then $\dim V \leq \dim W$ holds. Also, if G acts freely on $S(V)$ and $S(W)$, then if there exists a G-map $f : S(V) \to S(W)$, then $\dim V \leq \dim W$ holds.

Proposition 2.5. *Let $G = C_p^k$ or T^k and X a G-space. Then $cl_G(X) \leq l_G(X)$.*

Proof. Let $f : S(V) \to X$, $cl_G(X) = \dim V$ and $g : X \to S(W)$, $l_G(X) = \dim W$ be G-maps realizing the equivariant colevel and level respectively. We have a G-map $g \circ f : S(V) \to S(W)$. By the Borsuk-Ulam theorem, we obtain $cl_G(X) \le l_G(X)$. □

Assume that X is a G-representation sphere $S(V)$. Note that $l_G(S(V)) \le \dim V$ and $cl_G(S(V)) \ge \dim V$, since the identity map $id : S(V) \to S(V)$ is a G-map.

Proposition 2.6. *The following inequalities hold.*

(1) $l_G(S(V \oplus W)) \le l_G(S(V)) + l_G(S(W))$.

(2) $cl_G(S(V \oplus W)) \ge cl_G(S(V)) + cl_G(S(W))$.

Proof. (1) Let $l_G(S(V)) = k$ and $l_G(S(W)) = l$. Let $f : S(V) \to S(V')$, $k = \dim V'$ and $g : S(W) \to S(W')$, $l = \dim W'$ be G-maps realizing the levels k and l. Then $f * g : S(V \oplus W) \to S(V' \oplus W')$ is a G-map, where $*$ means join. This implies

$$l_G(S(V \oplus W)) \le k + l = l_G(S(V)) + l_G(S(W)).$$

(2) This is proved by a similar argument as (1). □

These facts lead us to the following result.

Proposition 2.7. *The following statements hold.*

(1) *Let $G = C_p^k$ or T^k. For any fixed-point-free G-representation V, it follows that $l_G(S(V)) = cl_G(S(V)) = \dim V$.*

(2) *If G acts freely on $S(W)$, then $cl_G(S(W)) = \dim W$.*

Proof. (1) Let $f : S(V) \to S(W)$ be a G-map. By the Borsuk-Ulam theorem, $\dim V \le \dim W$. This means $\dim V \le l_G(S(V))$. As mentioned above, since $\dim V \ge l_G(S(V))$, it follows that $l_G(S(V)) = \dim V$. Similarly one can see $cl_G(S(V)) = \dim V$.

(2) Let $f : S(V) \to S(W)$ be a G-map. Since G acts freely on $S(W)$, it follows that G acts freely on $S(V)$. Hence $\dim V \le \dim W$ holds by the Borsuk-Ulam theorem. This implies $cl_G(S(W)) = \dim W$. □

Remark. Even if G acts freely on $S(V)$, it is not necessary to hold $l_G(S(V)) = \dim V$. Such an example can be found in Theorem 3.3 in the next section. Furthermore, if G is neither C_p^k nor T^k, then there exists a G-representation V such that $cl_G(S(V)) > \dim V$ by results of [12], [13].

At the end of this section, we mention a relation of the equivariant level and the genus introduced by [1]. Let $\mathcal{S}_G$ be the set of closed proper subgroups of G. Let

$\mathrm{Iso}(X)$ be the set of isotropy subgroups of X, where X is a G-space with $X^G = \emptyset$. The $\mathcal{S}_G$-genus $\tilde{\gamma}_G(X)$ of X is defined by the minimal number k such that there exists a G-map $f : X \to *_{i=1}^k G/H_i$, $H_i \in \mathcal{S}_G$, where $*$ means join. Similarly the $\mathrm{Iso}(X)$-genus $\gamma_G(X)$ of X is defined by the minimal number k such that there exists a G-map $f : X \to *_{i=1}^k G/H_i$, $H_i \in \mathrm{Iso}(X)$. Clearly $\tilde{\gamma}_G(X) \le \gamma_G(X)$. We summarize general properties of genera of X. See [1] for more information.

Proposition 2.8. *The following statements hold.*

(1) *If there exists a G-map $f : X \to Y$, then $\tilde{\gamma}_G(X) \le \tilde{\gamma}_G(Y)$.*

(2) *If there exists a G-map $f : X \to Y$ and $\mathrm{Iso}(X) = \mathrm{Iso}(Y)$, then $\gamma_G(X) \le \gamma_G(Y)$.*

(3) *$\tilde{\gamma}_G(S(V \oplus W)) \le \tilde{\gamma}_G(S(V)) + \tilde{\gamma}_G(S(W))$ and $\gamma_G(S(V \oplus W)) \le \gamma_G(S(V)) + \gamma_G(S(W))$.*

Proof. (1) Let $\tilde{\gamma}_G(Y) = k$ and $g : Y \to *^k G/H_i$ a G-map realizing k. Considering a G-map $g \circ f : X \to *^k G/H_i$, one sees $\tilde{\gamma}_G(X) \le \tilde{\gamma}_G(Y)$.

(2) Let $\gamma_G(Y) = k$ and $h : X \to *^k G/H_i$, $H_i \in \mathrm{Iso}(Y)$, a G-map realizing k. Since $H_i \in \mathrm{Iso}(X)$, it follows that $\gamma_G(X) \le \gamma_G(Y)$.

(3) Let $\tilde{\gamma}_G(S(V)) = k$ and $\tilde{\gamma}_G(S(W)) = l$. Let $f : X \to *_{i=1}^k G/H_i$ and $g : X \to *_{j=1}^l G/K_j$ be G-maps realizing k and l respectively. Then there exists a G-map

$$f * g : S(V \oplus W) \cong S(V) * S(W) \to (*_{i=1}^k G/H_i) * (*_{j=1}^l G/K_j).$$

This implies the first inequality. Assume $H_i \in \mathrm{Iso}(S(V))$ and $K_j \in \mathrm{Iso}(S(W))$. Since $\mathrm{Iso}(S(V)) \cup \mathrm{Iso}(S(W)) \subset \mathrm{Iso}(S(V \oplus W))$, the second inequality holds. □

Assume hereafter that G is a compact abelian Lie group and $X = S(V)$ is a fixed-point-free representation sphere. Decompose V into $V = \oplus_K V(K)$, where $V(K)$ is the direct sum of irreducible sub-representations with kernel K. Let $I(V)$ be the set of K with $V(K) \ne 0$. Note that G/K is C_p (p: prime) or S^1 for $K \in I(V)$ and $I(V) \subset \mathrm{Iso}(S(V))$. Set $U_K = \mathrm{Inf}_{G/K}^G U_{\{1\}}$ for $K \in I(V)$, where $U_{\{1\}}$ is the standard G/K-representation. Note that $\dim U_K = 1$ if $|G/K| = 2$ and $\dim U_K = 2$ otherwise. Let U'_K be another irreducible G-representation with kernel K. Then there exists G-isovariant maps $f : S(U_K) \to S(U'_K)$ and $g : S(U'_K) \to S(U_K)$ by results of [9]. Therefore we may assume that $V(K)$ is a direct sum of U_K in computing $l_G(S(V))$, $cl_G(S(V))$, $\tilde{\gamma}_G(S(V))$ and $\gamma_G(S(V))$. We set $I_2(V) = \{K \in I(V) \,|\, G/K \cong C_2\}$ and $I'(V) = I(V) \smallsetminus I_2(V)$. We show the following.

Theorem 2.9. *Let G be abelian. Then $\tilde{\gamma}_G(S(V)) \le l_G(S(V)) \le 2\gamma_G(S(V))$.*

Proof. Let $\gamma_G(S(V)) = k$ and $g : S(V) \to *_{i=1}^k G/H_i$, $H_i \in \mathrm{Iso}(S(V))$ a G-map realizing $\tilde{\gamma}_G(S(V)) = k$. For any H_i, one can take $K_i \in I(V)$ such that $H_i \le K_i$. Clearly there exits a G-map $j : *_{i=1}^k G/H_i \to *_{i=1}^k G/K_i$ and there exists a G-map $f_i : G/K_i \to S(U_{K_i})$ and hence one obtains a G-map

$$h := *_{i=1}^k f_i : *_{i=1}^k G/K_i \to S(\oplus_{i=1}^k U_{K_i}).$$

Thus we have a G-map $h \circ j \circ g : S(V) \to S(\oplus_{i=1}^k U_{K_i})$. Since $\dim U_{K_i} \le 2$, it follows that $l_G(S(V)) \le 2\gamma_G(S(V))$.

We next show $l_G(S(V)) \ge \tilde{\gamma}_G(S(V))$. Let $l_G(S(V)) = l$ and $f : S(V) \to S(W)$, $\dim W = l$, a G-map realizing $l_G(S(V)) = l$. It it easy to see that there exists a G-map $f_K : S(U_K) \cong S^0 \to G/K \cong C_2$ for $K \in I_2(W)$ and there exists a G-map $g_L : S(U_L) \cong S^1 \to G/L * G/L$ for $L \in I'(W)$. Thus we obtain G-maps $h_K : S(W(K)) \to *^{w_K} G/K$ for $K \in I_2(W)$ and $h_L : S(W(L)) \to *^{v_L}(G/L * G/L)$ for $L \in I'(W)$, where $v_K = \dim W(K)$ and $v_L = \dim W(L)/2$. Therefore we obtain a G-map

$$h : S(W) \to *_{K \in I_2(W)}(*^{w_K} G/K) * *_{L \in I'(W)}(*^{v_L}(G/L * G/L)).$$

Composing a G-map h with f, we obtain that

$$\tilde{\gamma}_G(S(V)) \le \sum_{K \in I_2(W)} w_K + \sum_{L \in I'(W)} 2v_L = \dim W = l_G(S(V)).$$

□

Remark. In the above proof, if $I(W) \subset I(V)$, then $\gamma_G(S(V)) \le l_G(S(V))$ holds.

Example 2.10. Let $G = S^1$ and $V = U_{\{1\}}$. Then $\tilde{\gamma}_G(S(V)) = \gamma_G(S(V)) = 1$ and $l_G(S(V)) = 2$. In this case, $l_G(S(V)) = 2\gamma_G(S(V))$ holds.

More generally, the following result holds.

Proposition 2.11. *The following statements hold.*

(1) *If $G = T^k$, then $l_G(S(V)) = 2\tilde{\gamma}_G(S(V)) = 2\gamma_G(S(V)) = \dim V$.*
(2) *If $G = C_p^k$, then $l_G(S(V)) = \tilde{\gamma}_G(S(V)) = \gamma_G(S(V)) = \dim V$.*

Proof. By Proposition 2.7, we already know that $l_G(S(V)) = \dim V$ for $G = T^k$ and C_p^k. By results of [1], it is known that $\tilde{\gamma}_G(S(V)) = \gamma_G(S(V)) = (\dim V)/2$ when $G = T^k$ and $\tilde{\gamma}_G(S(V)) = \gamma_G(S(V)) = \dim V$ when $G = C_p^k$. Thus the desired result holds. □

3. The equivariant level and colevel of C_{pq}-representation spheres

Let G be a cyclic group C_{pq} of order pq, where p, q are distinct primes. In this section, we compute the equivariant level and colevel of G-representation spheres in several cases. We set $V = V(1) \oplus V(C_p) \oplus V(C_q)$. Set $U_k = \mathrm{Inf}_{G/C_k}^G U_{\{1\}}$ for $k = 1$, p or q, where $U_{\{1\}}$ is the standard irreducible $C_{pq/k}$-representation. Note that if p and q are odd primes, then $\dim U_k = 2$, and if $q = 2$, then $\dim U_k = 2$ for $k = 1$, 2, and $\dim U_p = 1$. We may assume that $V(C_k)$ is a direct sum of copies of U_k for $k = 1$, p, q as mentioned before.

We here consider the case where p and q are distinct primes. We discuss the results in several cases.

Theorem 3.1. *Let $G = C_{pq}$, where p and q are distinct primes with $p > q$ and V a fixed-point-free G-representation. Then*

(1) $l_G(S(V)) \geq \dim V(C_p) + \dim V(C_q) = \dim V^{C_p} + \dim V^{C_q}$.

(2) *If* $\dim V^{C_p} \geq 2$ *and* $\dim V^{C_q} \neq 0$, *then*

$$l_G(S(V)) = \dim V(C_p) + \dim V(C_q) = \dim V^{C_p} + \dim V^{C_q}.$$

(3) *If* $\dim V^{C_p} \geq 2$ *and* $\dim V^{C_q} \neq 0$, *then* $cl_G(S(V)) = \infty$.

Proof. (1) Let $f : S(V) \to S(W)$ be a G-map. Applying the Borsuk-Ulam theorem to a C_p-map $f^{C_q} : S(V)^{C_q} = S(V(C_q)) \to S(W)^{C_q} = S(W(C_q))$, one sees $\dim V(C_q) \leq \dim W(C_q)$. Similarly one sees $\dim V(C_p) \leq \dim W(C_p)$. Since $\dim W \geq \dim V(C_p) \oplus V(C_q)$, it follows that $l_G(S(V)) \geq \dim V(C_p) + \dim V(C_q)$.

(2) Since C_{pq}/C_q is of odd order, it follows that $\dim V(C_q) \geq 2$. Set $W = V(C_p) \oplus V(C_q)$ and consider the identity map

$$i : S(V(C_p) \oplus V(C_q)) \to S(V(C_p) \oplus V(C_q)).$$

By an obstruction theoretic argument of [15] or [10], i is extended to a G-map $g : S(V) \to S(V(C_p) \oplus V(C_q))$. Therefore $l_G(S(V)) \leq \dim V(C_p) + \dim V(C_q)$. Therefore (2) holds.

(3) Similarly there exits a G-map $g_n : S(nU_1 \oplus V(C_p) \oplus V(C_q)) \to S(V)$ for any $n \geq 1$. This implies that $cl_G(S(V)) = \infty$. □

Remark. By results of [2], if G is not a p-toral group, then there exists a G-representation V such that $cl_G(S(V)) = \infty$, and if G is a finite p-group, then $cl_G(S(V)) < \infty$.

Theorem 3.2. *Let $G = C_{pq}$, where p and q are distinct primes with $p > q$ and V a fixed-point-free G-representation. Assume that* $\dim V(C_p) = 0$ *or* $\dim V(C_q) = 0$. *Then $cl_G(S(V)) = \dim V$.*

Proof. We may suppose $V(C_q) = 0$, hence $V = V(1) \oplus V(C_p)$. Let $f : S(W) \to S(V)$ be a G-map. By the Borsuk-Ulam theorem, one has $\dim W(C_p) \leq \dim V(C_p)$ and $\dim W(C_q) = 0$. Thus $W = W(1) \oplus W(C_p)$. Since C_q acts freely on $S(W)$ and $S(V)$, it follows from the Borsuk-Ulam theorem that $\dim W \leq \dim V$. Thus $cl_G(S(V)) \leq \dim V$. On the other hand, clearly $cl_G(S(V)) \geq \dim V$ and therefore $cl_G(S(V)) = \dim V$. □

Theorem 3.3. *Let $G = C_{pq}$, where p and q are distinct primes with $p > q$ and V a fixed-point-free G-representation.*

(1) *Suppose that $V(C_p) = 0$, $V(C_q) \neq 0$. Then*
  (a) *If $V(1) \neq 0$ and $q \neq 2$, then $l_G(S(V)) = \dim V(C_q) + 2$.*
  (b) *If $V(1) \neq 0$ and $q = 2$, then $\dim V(C_q) + 1 \leq l_G(S(V)) \leq \dim V(C_q) + 2$.*
  (c) *If $V(1) = 0$, then $l_G(S(V)) = \dim V(C_q) = \dim V$.*

(2) *Suppose that $V(C_p) \neq 0$, $V(C_q) = 0$. Then*
  (a) *If $V(1) \neq 0$ and $\dim V(C_p) \geq 2$, then $l_G(S(V)) = \dim V(C_p) + 2$.*
  (b) *If $V(1) \neq 0$ and $\dim V(C_p) = 1$ (this happens only when $q = 2$), then $3 \leq l_G(S(V)) \leq 4$.*
  (c) *If $V(1) = 0$, then $l_G(S(V)) = \dim V(C_p) = \dim V$.*

(3) *Suppose that $V(C_p) = V(C_q) = 0$. Then*
  (a) *If q is an odd prime and $\dim V = 2$. then $l_G(S(V)) = 2$.*
  (b) *If q is an odd prime and $\dim V \geq 4$, then $l_G(S(V)) = 4$.*
  (c) *If $q = 2$, then $3 \leq l_G(S(V)) \leq 4$.*

Proof. (1) Suppose that $V = V(1) \oplus V(C_q)$. Let $f : S(V) \to S(W)$ be a G-map. By the Borsuk-Ulam theorem, one has $\dim V(C_q) \leq \dim W(C_q)$.

Set $U'_p = U_p$ for q is an odd prime, and $U'_p = 2U_p$ for $q = 2$. Thus $\dim U'_p = 2$. Set $W' = U'_p \oplus V(C_q)$. Then there exits a G-map $g : S(V) \to S(W')$ as before. Hence $l_G(S(V)) \leq \dim W' = \dim V(C_q) + 2$. By Theorem 3.1, $\dim V(C_q) \leq l_G(S(V))$. If $l_G(S(V)) = \dim V(C_q)$, then there exists a G-map $f : S(V) \to S(V(C_q))$, but this contradicts the Borsuk-Ulam theorem for a C_p-map. Therefore the desired results (a) and (b) hold.

(1-c) Since $V = V(C_p)$, it follows from Theorem 2.2 that $l_G(V) \leq l_{C_q}(V^{C_p}) = \dim V$. On the other hand, $l_G(V) \geq \dim V$ and therefore (c) holds.

(2) The proof is similar with (1).

(3) By a similar argument, one sees that $l_G(S(V)) \leq 4$. If $l_G(S(V) \leq 2$, then there are no G-maps when $\dim V \geq 4$ by the Borsuk-Ulam theorem. Therefore $3 \leq l_G(S(V)) \leq 4$. □

Remark. Let $G = C_{2p}$, where p is an odd prime. By a result of [11], if $V = 2U_1$, then $l_G(S(V)) = 3$.

In almost cases, we have determined the equivariant level and colevel for C_{pq}. The remaining cases are (1-b), (2-b) and (3-c) in Theorem 3.3. We would like to study these cases in future research.

Finally we discuss the equivariant level when $p = q$. In this case, this is essentially studied by [14] and [8]. We restate their results in our context. Set

$$L_p^{2m-1} := S(mU_1)/C_p,$$

where U_1 is the standard free C_{p^2}-representation. If $p = 2$, then L_2^{2m-1} is the $(2m-1)$-dimensional real projective space with the standard free C_2-action, and if p is an odd prime, then L_p^{2m-1} is the $(2m-1)$-dimensional lens space with the standard free C_p-action.

Lemma 3.4. *Let $G = C_{p^2}$. Then $l_G(S(mU_1)) = l_{C_p}(L_p^{2m-1})$.*

Proof. In the case of $p = 2$, i.e., $G = C_4$. We may set $V = V(1) \oplus V(C_2)$. Let $f : L_2^{2m-1} \to S(\mathbb{R}_\varepsilon^l)$ be a C_2-map realizing $l_{C_2}(L_2^{2m-1}) = l$, where $\mathbb{R}_\varepsilon^l$ is the nontrivial irreducible C_2-representation. Let $q : C_4 \to C_2$ be the projection and

$$\pi : S(mU_1) \to S(mU_1)/C_2 = L_2^{2m-1}$$

be the covering map which is a q-equivariant map. Also the identity map

$$i : S(lU_2) \to S(lU_2)/C_2 = S(l\mathbb{R}_\varepsilon)$$

is a q-equivariant map. Then $\tilde{f} := i^{-1} \circ f \circ \pi : S(mU_1) \to S(lU_2)$ is a G-map over f. Thus $l_G(S(mU_1)) \leq l = l_{C_2}(\mathbb{R}P^{2m-1})$.

Conversely, let $f : S(mU_1) \to S(W)$, $\dim W = l$, be a G-map realizing $l_G(S(V)) = l$. There exists a G-map $j : S(U_1) \to S(U_2 \oplus U_2)$, where $U_2 = \mathrm{Inf}_{G/C_2}^G \mathbb{R}_\varepsilon$. Hence we may suppose that $W = lU_2$. Then $\bar{f} : L_2^{2m-1} = S(mU_1)/C_2 \to S(lU_2)/C_2 = S(l\mathbb{R}_\varepsilon)$ is a C_2-map. Thus $l_{C_2}(L_2^{2m-1}) \leq l = l_G(S(V))$. Therefore, (1) holds.

When p is an odd prime, a similar argument leads to the formula. We omit the detail. □

The level $l_{C_2}(L_2^{2m-1})$ has been computed by [14] and $l_{C_p}(L_p^{2m-1})$, p: odd prime, by [8]. By Lemma 3.4, we obtain the following.

Proposition 3.5. *The following hold.*

(1) $l_{C_4}(S(mU_1)) = \begin{cases} m+1 & m \equiv 0, 2 \bmod 8 \\ m+2 & m \equiv 1, 3, 4, 5, 7 \bmod 8 \\ m+3 & m \equiv 6 \bmod 8. \end{cases}$

(2) *If p is an odd prime, then*

(a) $2\langle (m-2)/p \rangle + 2 \le l_{C_{p^2}}(S(mU_1)) \le 2\langle (m-2)/p \rangle + 4$ *for* $m \not\equiv 2 \bmod p$, *where* $\langle x \rangle$ *denotes the smallest integer more than or equal to* x.

(b) $l_{C_{p^2}}(S(mU_1)) = 2(m-2)/p + 4$ *for* $m \equiv 2 \bmod p$.

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References

[1] T. Bartsch, *On the genus of representation spheres.*, Comment. Math. Helv. 65 (1990),85–95.

[2] T. Bartsch, *On the existence of Borsuk-Ulam theorems*, Topology 31 (1992) 533–543.

[3] G. E. Bredon, Introduction to compact transformation groups, Academic Press, 1972.

[4] P. E. Conner and E. E. Floyd. *Fixed point free involutions and equivariant maps*, Bull. Amer. Math. Soc. 66 (1960), 416–441.

[5] P. E. Conner and E. E. Floyd. *Fixed point free involutions and equivariant maps II*, Trans. Amer. Math. Soc. 105 (1962), 222–228.

[6] T. tom Dieck, Transformation groups, *Transformation Groups*, Walter de Gruyter, Berlin, New York, 1987.

[7] A. Pfister and S. Stolz. On the level of projective spaces, Comm. Math. Helv. 62 (1987), 286–291.

[8] D. M. Meyer, *$\mathbb{Z}/p$-equivariant maps between lens spaces and spheres*, Math. Ann. 312 (1998), 197–214.

[9] I. Nagasaki, *On the isovariant mapping equivalence between representation spaces*, Studia Humana et Naturalia 45 (2011) 45–52.

[10] I. Nagasaki, *Remarks on equivariant and isovariant maps between representations*, Studia Humana et Naturalia 47 (2013) 61–67.

[11] I. Nagasaki, *Equivariant maps between C_{2p}-representation spheres for an odd prime p*, Studia Humana et Naturalia 54 (2020) 71–77.

[12] I. Nagasaki, *Elementary abelian p-groups are the only finite groups with the Borsuk-Ulam property*, J. Fixed Point Theory Appl. 21 (2019), Article 16.
[13] I. Nagasaki, *Determination of compact Lie groups with the Borsuk-Ulam property*, Top. Appl. 319 (2022), Article 108216.
[14] S. Stolz, *The level of real projective spaces*, Comment. Math. Helv. 64 (1989), 661–674.
[15] S. Waner, *A note on the existence of G-maps between spheres*, Proc. Amer. Math. Soc. 99 (1987), 179-181.