Remarks on Equivariant and Isovariant Maps between Representations

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Abstract. In this note, we consider the existence problem of equivariant or isovariant maps between representation spheres. In particular, we give a necessary and sufficient condition for the existence of an equivariant map between unitary representation spheres of a cyclic group C_{pq}, where p, q are distinct primes.

1. The existence problem of C_{pq}-maps

The existence or non-existence problem of equivariant maps is a fundamental and important topic in equivariant topology, and many results are known up to the present. However, giving a necessary and sufficient condition for the existence of an equivariant map is not so easy in general. Recently, Marzantowicz, de Mattos and dos Santos [6] discuss a necessary and sufficient condition of the existence of an equivariant map for a torus and a p-torus. In this note, we deal with the case of C_{pq}-maps, where p, q are distinct primes.

First, we recall well-known results on the existence problem. Let G be a compact Lie group and V an (orthogonal) representation of G. We denote by SV the representation sphere of V, which is defined as the unit sphere of V. The following fact is proved by equivariant obstruction theory; for example, see [2].

Proposition 1.1. *Let V and W be (orthogonal) representations of G. If $\dim V^H \le \dim W^H$ for every (closed) subgroup H of G, then there exists a G-map $f\colon SV \to SW$.*

The converse is not true in general, but in some special cases, the converse holds. Such kind of results are brought from Borsuk-Ulam type theorems. We state two Borsuk-Ulam type theorems; see [3], [4], [5], [11] for more details.

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Proposition 1.2. *Assume that G acts freely on SV and SW. If there exists a G-map $f\colon SV \to SW$, then* $\dim V \leq \dim W$ *holds.*

Proposition 1.3. *Let G be a torus T^n or a p-torus C_p^n. Assume that SV and SW are G-fixed point free, i.e., $SV^G = SW^G = \emptyset$. If there exists a G-map $f\colon SV \to SW$, then* $\dim V \leq \dim W$ *holds.*

Marzantowicz, de Mattos and dos Santos [6] give a necessary and sufficient condition for the existence of an equivariant map in the case of G being a torus or a p-torus. In particular, the following result is deduced from their results.

Corollary 1.4. *Let G be a torus T^n or a p-torus C_p^n. Let SV and SW be G-fixed point free representation spheres. Then there exists a G-map $f\colon SV \to SW$ if and only if* $\dim V^H \leq \dim W^H$ *holds for every closed subgroups H of G.*

Proof. Note that G/H is a torus or a p-torus; in fact, if $G = T^n$, then G/H is connected and abelian. Hence G/H is a torus. If $G = C_p^n$, then G/H is an abelian group consisting of elements of order p or 1. Hence G/H is a p-torus. In each case, $f^H : SV^H \to SW^H$ is a G/H-map between G/H-fixed point free representation spheres. Hence Proposition 1.3 shows the necessary condition. The sufficient condition follows from Proposition 1.1. □

We now consider equivariant maps between representation spheres of a cyclic group C_{pq}, where p, q are distinct primes. Let $G = C_{pq}$ and c a generator of C_{pq}. The unitary irreducible representations U_k $(0 \leq k \leq pq-1)$ of C_{pq} are given by

$$cz = \xi^k z \quad \text{for} \quad z \in U_k = \mathbb{C},\ \xi = \exp\frac{2\pi\sqrt{-1}}{pq}.$$

Each orthogonal irreducible representation T_k is given as the following way: $T_0 = \mathbb{R}$ with the trivial action; if $0 < k < pq/2$, then $T_k = r_{\mathbb{R}} U_k$, where $r_{\mathbb{R}}$ denote realification of a unitary representation, and if $q = 2$ and p is an odd prime, then $T_p = \mathbb{R}_-$ with the antipodal action of C_2 and the trivial action of C_p.

Set $C_p = \langle c^q \rangle$ and $C_q = \langle c^p \rangle$. Let V and W be orthogonal representations with $V^G = W^G = 0$. If there exists a G-map $f\colon SV \to SW$, then $f^H\colon SV^H \to SW^H$ is a G/H-map for $H = C_p$ or C_q. Since G/H acts freely on SV^H and SW^H, it follows from Proposition 1.2 that $\dim V^H \leq \dim W^H$ for $H = C_p$, C_q. In the case of $\dim W^H = 0$ $(H = C_p$ or $C_q)$, it follows that $\dim V^H = 0$. Since $\operatorname{res}_H f$ is an H-map between H-free representation spheres, we have $\dim V \leq \dim W$ by Proposition 1.2. Thus we obtain the following.

Proposition 1.5. *Let $G = C_{pq}$. Let V and W be representations with $V^G = W^G = 0$. If there exists a G-map $f\colon SV \to SW$, then the following hold.*

(1) $\dim V^{C_p} \leq \dim W^{C_p}$ *and* $\dim V^{C_q} \leq \dim W^{C_q}$.

(2) *If* $\dim W^{C_p} = 0$ *or* $\dim W^{C_q} = 0$, *then* $\dim V \leq \dim W$.

In the next section, we show that if V and W are unitary, then the converse holds. As a consequence, we obtain the following.

Theorem 1.6. *Let V and W be unitary representations with $V^G = W^G = 0$ for $G = C_{pq}$. There exists a G-map $f\colon SV \to SW$ if and only if the conditions* (1) *and* (2) *of Proposition* 1.5 *hold.*

2. Proof of Theorem 1.6

We have already shown that the conditions (1) and (2) are necessary. Next we show that (1) and (2) are sufficient for the existence of a G-map. The proof is divided into several cases.

We set $G = C_{pq}$ and denote by U_k the unitary irreducible representation of C_{pq} described in the previous section. The following is straightforward.

Lemma 2.1. *If $f_i\colon SV_i \to SW_i$, $i = 1, 2$, are G-maps, then the join of f_1 and f_2 induces a G-map $f_1 * f_2\colon S(V_1 \oplus V_2) \to S(W_1 \oplus W_2)$.*

The kernel $\operatorname{Ker} V$ of a representation V is defined by the kernel of the representation homomorphism of V: $\rho_V\colon G \to GL(V)$. It is easily seen that

$$\operatorname{Ker} U_k = \begin{cases} 1 & (k, pq) = 1 \\ C_p & (k, pq) = p \\ C_q & (k, pq) = q \\ C_{pq} & k = 0, \end{cases}$$

where (k, pq) denotes the greatest common divisor of k and pq.

Lemma 2.2. *If $\operatorname{Ker} U_k = \operatorname{Ker} U_l$, then there exists a G-map $f : SU_k \to SU_l$,*

Proof. If $\operatorname{Ker} U_k = \operatorname{Ker} U_l = C_{pq}$, then it is trivial. Assume that $\operatorname{Ker} U_k = \operatorname{Ker} U_l \neq C_{pq}$ Since $(k/(k, pq), pq) = 1$, one can take an integer s such that $sk/(k, pq) \equiv 1 \pmod{pq}$.

Then a map f defined by

$$f(z) = z^{sl/(k,pq)},\ z \in SU_k$$

is a G-map. □

For a representation V with $V^G = 0$, decompose V into irreducible representations as follows:

$$V = \bigoplus_{i=1}^{pq-1} a_i U_i\ (a_i \geq 0).$$

Let H be a subgroup of G. Setting $V(H) = \bigoplus_{i:\mathrm{Ker}\, U_i = H} a_i U_i$, we have a decomposition

$$V = V(1) \oplus V(C_p) \oplus V(C_q).$$

By Lemmas 2.1 and 2.2, we obtain the following.

Proposition 2.3. *There exist G-maps between $S(V(H))$ and $S(mU_{(|H|,pq)})$ bidirectionally, where $m = \frac{1}{2}\dim V(H)$.*

By this proposition, without loss of generality, we may assume that V and W have the following forms:

$$V = a_1 U_1 \oplus a_p U_p \oplus a_q U_q,$$
$$W = b_1 U_1 \oplus b_p U_p \oplus b_q U_q,$$

where a_i and b_i are non-negative integers. Note that $V^{C_p} = a_p U_p$, $V^{C_q} = a_q U_q$ and so on. It is easy to see that the conditions (1) and (2) are equivalent to statements:

(1) $a_p \leq b_p$ and $a_q \leq b_q$.

(2) If $b_p = 0$, then $a_1 + a_q \leq b_1 + b_q$ and if $b_q = 0$, then $a_1 + a_p \leq b_1 + b_p$.

First we recall the following result from [12].

Lemma 2.4. *Let $W = b_1 U_1 \oplus b_p U_p \oplus b_q U_q$, $b_p > 0$, $b_q > 0$. Then there exists a self G-map $h\colon SW \to SW$ such that* $\deg h = 0$.

Proof. Degrees of h^H, $H \leq G$, of a self G-map h on SW satisfy the Burnside ring relation described in [2]. In fact, it is seen that if there exists a G-map $h\colon SW \to SW$, then the following relations hold:

$$\begin{cases} \deg h \equiv \deg h^{C_p} \bmod p, \\ \deg h \equiv \deg h^{C_q} \bmod q, \\ \deg h^{C_p} \equiv 1 \bmod q, \\ \deg h^{C_q} \equiv 1 \bmod p. \end{cases}$$

Conversely, if integers d_1, d_p, d_q satisfy relations $d_1 \equiv d_p \bmod p$, $d_1 \equiv d_q \bmod q$, $d_p \equiv 1 \bmod q$ and $d_q \equiv 1 \bmod p$, then there exists a G-map $h\colon SW \to SW$ such that $\deg h = d_1$, $\deg h^{C_p} = d_p$, $\deg h^{C_q} = d_q$. We set $d_1 = 0$ and we can take d_p such that $d_p \equiv 0 \bmod p$, $d_p \equiv 1 \bmod q$ and d_q such that $d_q \equiv 1 \bmod p$, $d_q \equiv 0 \bmod q$, These integers satisfy the above relations. Therefore there exists a G-map $h\colon SW \to SW$ such that $\deg h = 0$. □

2.1. **Case 1.** We shall show the theorem when $b_p > 0$ and $b_q > 0$.

Lemma 2.5. *If* (1) $a_p \leq b_p$, $a_q \leq b_q$ *and* (2) $b_p > 0$, $b_q > 0$, *then there exists a G-map* $f\colon SV \to SW$.

Proof. By (1), there is an inclusion $i\colon SV^{>1} \to SW^{>1} \subset SW$. Using Waner's method [12], we show that this inclusion can be extended to a G-map f. Since G acts freely on $SV \smallsetminus SV^{>1}$, SV is decomposed as a union of $SV^{>1}$ and free G-cells:

$$SV = SV^{>1} \cup G \times D^{n_1} \cup \cdots \cup G \times D^{n_r},$$

where $n_1 \leq \cdots \leq n_r$. Set $X_k = SV^{>1} \cup G \times D^{n_1} \cup \cdots \cup G \times D^{n_k}$, where $k \geq 1$. Suppose inductively that there is a G-map $f_{k-1}\colon X_{k-1} \to SW$, where $X_0 = SV^{>1}$ and $f_0 = i$. Since $X_k = X_{k-1} \cup G \times D^{n_k}$, restricting f_{k-1} to $1 \times \partial D^{n_k} = \partial D^{n_k}$, we have a map $g = f_{k-1}|_{\partial D^{n_k}}\colon \partial D^{n_k} \to SW$. Compose g with a G-map h of degree 0 and set $g' = h \circ g$. Then g' is null-homotopic and g' is extended to $g''\colon D^{n_k} \to SW$. Furthermore, g'' is equivariantly extended to a G-map $\tilde{g}\colon G \times D^{n_k} \to SW$. Thus we obtain a G-map $f_k = f_{k-1} \cup \tilde{g}\colon X_k \to SW$. □

2.2. **Case 2.** We shall show the theorem in the case of $b_p = 0$ or $b_q = 0$. We may suppose $b_q = 0$. Then by condition (1), we have $a_q = 0$ and $a_p \leq b_p$, If $a_1 \leq b_1$, then, there is an inclusion $i\colon SV \to SW$, which is G-equivariant.

Suppose that $a_1 > b_1$. By condition (2), we have $a_1 + a_p \leq b_1 + b_p$ and hence $a_1 - b_1 \leq b_p - a_p$. Note that there exists a G-map $g\colon SU_1 \to SU_p$; for example, g can be defined by $g(z) = z^p$. Hence there exists a G-map $\bar{g}\colon S((a_1-b_1)U_1) \to S((b_p-a_p)U_p)$ by Lemma 2.1. Joining $\bar{g}$ with the identity map $id\colon S(b_1U_1 \oplus a_pU_p) \to S(b_1U_1 \oplus a_pU_p)$, we obtain a G-map $f = \bar{g} * id\colon SV \to SW$. Thus the proof is complete.

3. Comparison to isovariant maps

Let G be a compact Lie group. A continuous G-map $f\colon X \to Y$ is called G-isovariant if f preserves the isotropy groups; i.e., $G_{f(x)} = G_x$ for all $x \in X$. It is important to clarify the existence problem of isovariant maps and there are several researches about isovariant map as well as equivariant maps; for example, see [7], [8], [9], [10]. A necessary and sufficient condition for the existence of a C_{pq}-isovariant map between representations (or equivalently, representation spheres) is already known. In fact, the following result easily follows from results of [7].

Proposition 3.1. *Let $G = C_{pq}$. Let SV and SW be G-fixed point free (orthogonal) representation spheres. There exits a G-isovariant map $f\colon SV \to SW$ if and only if*

(1) $\dim V^H \le \dim W^H$ *for* $H = C_p$ *and* C_q, *and*

(2) $\dim V - \dim V^H \le \dim W - \dim W^H$ *for* $H = C_p$ *and* C_q.

Remark. By combining (1) and (2), it is deduced that $\dim V \le \dim W$. This kind of result is called the isovariant Borsuk-Ulam theorem. See [10], [13] for more general results.

By comparing Proposition 3.1 and Theorem 1.6, we see that there are many pairs SV, SW of C_{pq}-fixed point free representation spheres such that there is a C_{pq}-map from SV to SW, but not a C_{pq}-isovariant map.

Example 3.2. Let $V = aU_1$ ($a \ge 1$) and $W = U_p \oplus U_q$. Then there is a C_{pq}-map $f\colon SV \to SW$. However, if $a \ge 2$, then there is no C_{pq}-isovariant map from SV to SW.

This example provides another kind of Borsuk-Ulam type result; namely, if $a \ge 2$ and $f\colon SV \to SW$ is a C_{pq}-map, then it follows that $f^{-1}(SW^{>1}) \ne \emptyset$, where $SW^{>1}$ denotes the singular set of SW defined by $SW^{>1} = \{x \in SW \mid G_x \ne 1\}$. In this case, note that $SW^{>1} = SU_p \coprod SU_q$ (Hopf link). Furthermore, one can show the following.

Proposition 3.3. *Let $V = aU_1$ ($a \ge 2$) and $W = U_p \oplus U_q$. If there is a C_{pq}-map $f : SV \to SW$, then $f^{-1}(SU_p) \ne \emptyset$ and $f^{-1}(SU_q) \ne \emptyset$.*

Proof. If $f^{-1}(SU_p) = \emptyset$, then we have an equivariant map $f\colon SV \to SW \smallsetminus SU_p$. Since $SW \smallsetminus SU_p$ is C_{pq}-homotopy equivalent to SU_q, there exists a C_{pq}-map $g\colon SV \to SU_q$. However, this contradicts Proposition 1.5 (2). Thus we see that $f^{-1}(SU_p) \ne \emptyset$. Similarly we see that $f^{-1}(SU_q) \ne \emptyset$. □

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