A Note on the Weak Isovariant Borsuk-Ulam Theorem

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Abstract. In this note, we provide a refinement on the weak isovariant Borsuk-Ulam theorem. The main result is the following: For an arbitrary compact Lie group G, there exists a positive constant $c_G > 0$ such that, for any G-representations V and W, if there exists a G-isovariant map $f\colon V \to W$, then the inequality $c_G(\dim V - \dim V^G) \le \dim W - \dim W^G$ holds.

1. The weak isovariant Borsuk-Ulam theorem

Let G be a compact Lie group. A G-equivariant map $f\colon X \to Y$ is called G-*isovariant* if f preserves the isotropy subgroups: $G_{f(x)} = G_x$, $x \in X$. In this note, all maps between spaces are continuous.

In [3], we have shown the following Borsuk-Ulam type result.

Theorem 1.1. *For an arbitrary compact Lie group G, there exists a weakly monotonely increasing function*

$$\varphi_G\colon \mathbb{N}_0 \to \mathbb{N}_0 \ (\mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

diverging to ∞ with the following property:

(P) *For any G-representations V and W, if there is a G-isovariant map $f\colon V \to W$, then*

$$\varphi_G(\dim V - \dim V^G) \le \dim W - \dim W^G$$

holds.

However, a concrete form of φ_G is not given in [3]. In this note, we show that φ_G can be taken as a linear function $\varphi_G(n) = cn$ for some positive constant c. Namely, we show the following.

Dedicated to Professor Masakazu Tanatsugu on the occasion of his retirement from Kyoto Prefectural University of Medicine.

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Theorem 1.2. *For an arbitrary compact Lie group G, there exists a positive constant $c > 0$ such that, for any G-representations V and W, if there exists a G-isovariant map $f\colon V \to W$, then*

$$c\,(\dim V - \dim V^G) \leq \dim W - \dim W^G$$

holds.

For a nontrivial compact Lie group G, we define $\mathcal{C}_G$ as the set of $c \in \mathbb{R}$ such that

$$c\,(\dim V - \dim V^G) \leq \dim W - \dim W^G \tag{1.1}$$

holds for all G-representations V and W with a G-isovariant map $f\colon V \to W$.

Definition. We set $c_G = \sup \mathcal{C}_G$. For convenience, we set $c_G = 1$ when G is trivial.

Lemma 1.3. *The following hold.*

(1) $0 \in \mathcal{C}_G$.
(2) *If* $c' \leq c$ *and* $c \in \mathcal{C}_G$, *then* $c' \in \mathcal{C}_G$.
(3) $0 \leq c_G \leq 1$.

Proof. (1) When $c = 0$, the inequality (1.1) clearly holds.

(2) This follows by $c'(\dim V - \dim V^G) \leq c\,(\dim V - \dim V^G) \leq \dim W - \dim W^G$.

(3) By (1), $0 \leq c_G$. Taking the identity map $id\colon V \to V$ for a nontrivial representation, we see that $c \leq 1$ by (1.1). Hence $c_G \leq 1$. □

Lemma 1.4. *The supremum c_G is in the set $\mathcal{C}_G$; i.e., c_G is the maximum of c satisfying inequality* (1.1).

Proof. If $c_G = 0$, then $c_G \in \mathcal{C}_G$ by Lemma 1.3 (1). Suppose that $c_G > 0$. For any $\varepsilon > 0$, $c_G - \varepsilon \in \mathcal{C}_G$ by Lemma 1.3 (2). Then we have

$$(c_G - \varepsilon)(\dim V - \dim V^G) \leq \dim W - \dim W^G \quad \text{for any } \varepsilon > 0.$$

This implies that $c_G(\dim V - \dim V^G) \leq \dim W - \dim W^G$. □

Clearly, if $c_G = 1$ if and only if the isovariant Borsuk-Ulam theorem holds for G-representations; namely,

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds for all G-representations V and W with a G-isovariant map $f\colon V \to W$. The isovariant Borsuk-Ulam theorem was studied by Wasserman [6] and Nagasaki-Ushitaki

[5]. As a sufficient condition for $c_G = 1$, Wasserman gave the prime condition and Nagasaki and Ushitaki gave the Möbius condition. (For the details, see [5] and [6].) For example, from these conditions, one can see the following.

Proposition 1.5. *For the following groups,* $c_G = 1$*.*

(1) *Solvable compact Lie group.*
(2) *Alternating group* A_n *for* $5 \leq n \leq 11$*.*
(3) $\mathrm{PSL}(2, q)$*,* q *is a prime power.*

Remark. Some researchers conjecture that $c_G = 1$ for any finite group; however, it is still an open question.

2. Property of weak Borsuk-Ulam groups

We say that G is a *weak Borsuk-Ulam group* (weak BUG) if $c_G > 0$, and that G is a *Borsuk-Ulam group* (BUG) if $c_G = 1$. Theorem 1.2 implies that every compact Lie group G is a weak Borsuk-Ulam group, and Proposition 1.5 provides many examples of Borsuk-Ulam groups. We prepare several lemmas for showing Theorem 1.2.

Lemma 2.1. *Let* H *is a closed normal subgroup of* G*. If* H *and* G/H *is [weak] Borsuk-Ulam groups, then* G *is also a [weak] Borsuk-Ulam group.*

Proof. In the case where G and G/H are Borsuk-Ulam groups, Wasserman [6] has already shown it. Assume that G and G/H are weak Borsuk-Ulam groups. Let $f \colon V \to W$ be any G-isovariant map between representations. By restricting the action, $\mathrm{res}_H f \colon V \to W$ is H-isovariant and by fixing by H, $f^H \colon V^H \to W^H$ is G/H-isovariant. By definition, we have

$$c_H(\dim V - \dim V^H) \leq \dim W - \dim W^H$$

and

$$c_{G/H}(\dim V^H - \dim V^G) \leq \dim W^H - \dim W^G.$$

By adding the inequalities, we obtain

$$c_H(\dim V - \dim V^H) + c_{G/H}(\dim V^H - \dim V^G) \leq \dim W - \dim W^G.$$

Setting

$$c = \min\{c_H, c_{G/H}\} > 0,$$

we obtain

$$c\,(\dim V - \dim V^G) \leq \dim W - \dim W^G.$$

This shows that $c_G > 0$. □

Lemma 2.2. *Let H be a closed normal subgroup of G. Then if G is a [weak] Borsuk-Ulam group, then G/H is also a [weak] Borsuk-Ulam group.*

Proof. In the case where G is a Borsuk-Ulam group, Wasserman [6] has already shown it. Assume that G is a weak Borsuk-Ulam group. Let $f\colon V \to W$ be a G/H-isovariant map between G/H-representations. By using the projection $p\colon G \to G/H$, V and W are thought of as G-representations and f is thought of as a G-isovariant map. Then, we have

$$c_G(\dim V - V^G) \le \dim W - \dim W^G.$$

Since $\dim V^G = \dim V^{G/H}$ and $\dim W^G = \dim W^{G/H}$, we have

$$c_G(\dim V - V^{G/H}) \le \dim W - \dim W^{G/H}.$$

This means that $0 < c_G \le c_{G/H}$. □

Lemma 2.3. *Let H be a closed subgroup of G with $c_H = 1$. Assume that there exists a constant $0 < d < 1$ such that $\dim V^H \le d \dim V$ for all nontrivial irreducible G-representations V. Then $c_G \ge 1 - d > 0$. In particular, G is a weak Borsuk-Ulam group.*

Proof. Let $f\colon V \to W$ be a G-isovariant map between G-representations. Decompose $V = V^\perp \oplus V^G$ and $W = W^\perp \oplus W^G$, where $V^\perp$ [resp. $W^\perp$] denotes the orthogonal complement of V^G [resp. W^G]. Composing the natural inclusion $i\colon V^\perp \to V$ and the projection $p\colon W \to W^\perp$ with f, we obtain a G-isovariant map $g := p \circ f \circ i\colon V^\perp \to W^\perp$. Since H is a Borsuk-Ulam group, it follows that

$$\dim V^\perp - \dim {V^\perp}^H \le \dim W^\perp - \dim {W^\perp}^H \le \dim W^\perp.$$

By the complete reducibility of G, $V^\perp$ decomposes into a direct sum of nontrivial irreducible representations. Hence by assumption one can see that

$$(1-d) \dim V^\perp \le \dim V^\perp - \dim {V^\perp}^H.$$

Setting $c = 1 - d$, we obtain that $c \dim V^\perp \le \dim W^\perp$, or equivalently

$$c\,(\dim V - \dim V^G) \le \dim W - \dim W^G.$$

This implies that $c_G \ge c = 1 - d > 0$. □

Proposition 2.4. *A connected compact Lie group G is a weak Borsuk-Ulam group.*

Proof. Let T be a maximal torus of G. The following fact is proved in [3]. There exists a constant $0 < d < 1$ such that such that $\dim V^T \leq d \dim V$ for all nontrivial irreducible G-representations V. Therefore, by Lemma 2.3, a connected compact Lie group G is a weak Borsuk-Ulam group. $\square$

3. **Proof of Theorem 1.2 and some examples**

Let G be a (general) compact Lie group. Then there is a short exact sequence

$$1 \to G_0 \to G \to F \to 1,$$

where G_0 is the identity component of G and $F \cong G/G_0$ is a finite group. It follows from Proposition 2.4 that G_0 is a weak Borsuk-Ulam group. By Lemma 2.1, it suffices to show that a finite group is a weak Borsuk-Ulam group. By finite group theory, for every finite group G, there is a composition series

$$1 = H_0 \lhd H_1 \lhd \cdots \lhd H_r = G,$$

where each H_i/H_{i-1} is simple. Therefore, by using Lemma 2.1 repeatedly, it suffices to show that every finite simple group is a weak Borsuk-Ulam group. In [3], the following fact is proved.

Lemma 3.1. *Let G be a finite simple group. Let H be any nontrivial subgroup of G. Then there exists a constant $0 < d < 1$ such that $\dim V^H \leq d \dim V$ for all nontrivial irreducible representations V.*

In particular, if we take H as a nontrivial cyclic subgroup C, then since $c_C = 1$, it follows from Lemma 2.3 that every finite simple group is a weak Borsuk-Ulam group. As a consequence, an arbitrary finite group G is a weak Borsuk-Ulam group. Thus we obtain that every compact Lie group is a weak Borsuk-Ulam group.

Finally we give an example.

Proposition 3.2. *Let G be $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$. Then $c_G \geq 4/5$.*

Proof. The fact $c_{\mathrm{SO}(3)} \geq 4/5$ has already shown in [4]. Indeed, if we take $\mathrm{O}(2) \subset \mathrm{SO}(3)$, then $\dim V^{\mathrm{O}(2)} \leq \frac{1}{5} \dim V$ for every nontrivial irreducible representation of $\mathrm{SO}(3)$. Since $\mathrm{O}(2)$ is solvable, $c_{\mathrm{O}(2)} = 1$. By Lemma 2.3, we have $c_{\mathrm{SO}(3)} \geq 4/5$.

Next we consider the case of $\mathrm{SU}(2)$. Let

$$T = \left\{ g_t := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{C}, |t| = 1 \right\}$$

be a maximal torus of SU(2). Let N be the normalizer of T in SU(2), which is isomorphic to Pin(2). Indeed, one can see

$$N = T \coprod Tb, \quad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

By representation theory (see [2] for example), SU(2) has one k-dimensional complex irreducible representation V_k for each $k \geq 1$. Let χ_k be the character of V_k. By the Weyl character formula,

$$\chi_k(g_t) = \begin{cases} t^{k-1} + t^{k-3} + \cdots + t^2 + 1 + t^{-2} + t^{-5} + \cdots + t^{-(k-1)} & \text{if } k \text{ is odd} \\ t^{k-1} + t^{k-3} + \cdots + t + t^{-1} + t^{-3} + \cdots + t^{-(k-1)} & \text{if } k \text{ is even} \end{cases}$$

for $g_t \in T$. Note that

$$\begin{aligned} \dim_{\mathbb{C}} V_k^T &= \int_T \chi_k(g_t) dt \\ &= \begin{cases} 1 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

On the other hand, the order of $g_t b$ is 4 and $(g_t b)^2 = -I = g_{-1} \in T$. Since $g_t b$ is conjugate to g_i, we have

$$\chi_k(g_t b) = \chi_k(g_i) = \begin{cases} 1 & \text{if } k \equiv 1 \ (4) \\ -1 & \text{if } k \equiv -1 \ (4) \\ 0 & \text{otherwise} \end{cases}$$

and so

$$\begin{aligned} \dim_{\mathbb{C}} V_k^N &= \frac{1}{2}\left(\int_T \chi_k(g_t) dt + \int_{Tb} \chi_k(g_t b) dt \right) \\ &= \begin{cases} 1 & \text{if } k \equiv 1 \ (4) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus we have $\dim_{\mathbb{C}} V_k^N \leq \frac{1}{5} \dim_{\mathbb{C}} V_k$ for any k and so $\dim_{\mathbb{C}} V^N \leq \frac{1}{5} \dim_{\mathbb{C}} V$ for any complex representation V with $V^{\mathrm{SU}(2)} = 0$. For a real representation U, $U \oplus U$ has a complex structure. Then clearly $\dim_{\mathbb{C}} U \oplus U = \dim U$. Therefore we see that $\dim U^N \leq \frac{1}{5} \dim U$ for any real representation U with $U^{\mathrm{SU}(2)} = 0$. Since N is solvable, $c_N = 1$. By Lemma 2.3, we have $c_{\mathrm{SU}(2)} \geq 4/5$. □

Remark. Determining the precise value of $c_{\mathrm{SU}(2)}$ or $c_{\mathrm{SO}(3)}$ is still an open question.

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