Zero weight spaces of irreducible representations and a new estimate of the isovariant Borsuk-Ulam constant for SU(3)

Ikumitsu NAGASAKI [1)]

Abstract. Let G be the special unitary group SU(3) and T a maximal torus of G. For an irreducible G-representation V, the zero weight space, that is, the T-fixed point space V^T is considered as a representation of the Weyl group W of G. In this paper, we first determine the dimension of the W-fixed space $(V^T)^W$. As an application, we then provide a new estimate of the isovariant Borsuk-Ulam constant c_G. Indeed, we prove that $26/27 \le c_G \le 1$ for $G = \mathrm{SU}(3)$; this is a better estimate than our previous one.

1. Introduction

Let G be a compact Lie group. Let V and W be (orthogonal) G-representations and denote by $S(V)$ and $S(W)$ their unit spheres, called G-representation spheres. A G-map $f : S(V) \to S(W)$ is called *isovariant* if it preserves the isotropy groups.

The isovariant Borsuk-Ulam constant c_G is defined to be the supremum of a constant $c \in \mathbb{R}$ such that

$$c(\dim V - \dim V^G) \le \dim W - \dim W^G$$

holds whenever there exists a G-isovariant map $f : S(V) \to S(W)$. Obviously, $0 \le c_G \le 1$. The determination of c_G is an interesting and important problem for the study of isovariant Borsuk-Ulam type theorems. Especially, a compact Lie group G with $c_G = 1$ is called a *Borsuk-Ulam group.* Wasserman [6] and Nagasaki-Ushitaki [5] gave several examples of Borsuk-Ulam groups; in particular, any solvable compact Lie group G is a Borsuk-Ulam group. However, a complete classification of Borsuk-Ulam groups still remains as an open problem as well as the determination of c_G. Thus the estimation of the isovariant Borsuk-Ulam constant seems to be significant in order to approach this problem.

[1)] Department of Mathematics, Kyoto Prefectural University of Medicine, 1-5 Shimogamohangi-cho, Sakyo-ku, Kyoto 606-0823, Japan. e-mail: nagasaki@koto.kpu-m.ac.jp

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For any simple compact Lie group G with a maximal torus T, we define

$$d_G = \sup_V \{\dim V^T / \dim V\},$$

where V is taken over all nontrivial irreducible G-representations. In [3], we have proved that $c_G \geq 1 - d_G$ for any simple compact Lie group G and have also determined the values of d_G. In particular, we obtain $d_G = 1/(n+1)$ for $G = \mathrm{SU}(n)$; hence $c_{\mathrm{SU}(n)} \geq n/(n+1)$. Thus, $c_{\mathrm{SU}(2)} \geq 2/3$ and $c_{\mathrm{SU}(3)} \geq 3/4$. However, these estimates are not best possible. In fact, in [4], we have proved a better estimate $c_{\mathrm{SU}(2)} \geq 4/5$. In this paper, we shall show the following new estimate of $c_{\mathrm{SU}(3)}$.

Theorem 1.1. $c_{\mathrm{SU}(3)} \geq 26/27$.

Representation theory plays an important role in the proof of this result; in particular, the Weyl group fixed spaces of the zero weight representations of irreducible representations of SU(3) are used. In this paper, we first provide a complete computation of the dimension of such Weyl group fixed spaces and then prove the theorem.

2. Basic facts from representation theory

In order to prove our theorem, we here recall necessary basic facts from representation theory. Let $\widetilde{G}$ be the unitary group:

$$\widetilde{G} = \mathrm{U}(3) = \{A \in M_3(\mathbb{C}) \mid A^*A = E\}.$$

and G the special unitary group:

$$G = \mathrm{SU}(3) = \{A \in M_3(\mathbb{C}) \mid A^*A = E,\ \det A = 1\}.$$

A maximal torus $\widetilde{T}$ of $\widetilde{G}$ is given by

$$\widetilde{T} = \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \;\middle|\; |t_i| = 1 \right\} \cong S^1 \times S^1 \times S^1.$$

A maximal torus T of G is given by

$$T = \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \;\middle|\; t_1t_2t_3 = 1,\ |t_i| = 1 \right\} \cong S^1 \times S^1.$$

Let $\widetilde{W} = N_{\widetilde{G}}(\widetilde{T})/\widetilde{T} \cong S_3$ be the Weyl group of $\widetilde{G}$ and $W = N_G(T)/T \cong S_3$ the Weyl group of G. Taking the subgroup W_1 of all permutation matrices in $N_{\widetilde{G}}(\widetilde{T})$, we obtain $N_{\widetilde{G}}(\widetilde{T}) = \widetilde{T} \rtimes W_1$ and $\widetilde{W} \cong W_1$. Since there is a natural inclusion

$$i : N_G(T) \subset N_{\widetilde{G}}(T) = N_{\widetilde{G}}(\widetilde{T}),$$

we have an isomorphism $\iota : W \to W_1$ such that the following diagram commutes, see [1].

$$\begin{array}{ccc} G \supset N_G(T) & \xrightarrow{\text{proj}} & W \\ {\scriptstyle i}\big\downarrow & & {\scriptstyle \iota}\big\downarrow{\scriptstyle \cong} \\ \widetilde{G} \supset N_{\widetilde{G}}(\widetilde{T}) & \xrightarrow{\text{proj}} & W_1 \end{array}$$

In particular, ι^{-1} is given as follows. Let $w \in W_1$ be a permutation matrix and $\sigma_w \in S_3$ the corresponding permutation. Set

$$a(w) = \begin{pmatrix} \mathrm{sgn}(\sigma_w) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \widetilde{G}.$$

Then it follows that $\iota^{-1}(w) = a(w)wT \in W$.

Let $\widetilde{V} = \widetilde{V}(\lambda)$ be the irreducible (unitary) $\widetilde{G}$-representation with highest weight

$$\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda,$$

where Λ is the weight lattice consisting of the nonincreasing sequences $(\lambda_1, \lambda_2, \lambda_3)$ of integers, that is,

$$\Lambda = \{\lambda \in \mathbb{Z}^3 \mid \lambda_1 \geq \lambda_2 \geq \lambda_3\}.$$

It is well known that all irreducible $\widetilde{G}$-representations are parametrized by Λ, see for example [2]. Furthermore any irreducible G-representation V is obtained by the restriction to G of some irreducible $\widetilde{G}$-representation $\widetilde{V}(\lambda)$. We set $V(\lambda) := \mathrm{res}_G \widetilde{V}(\lambda)$. Then the following is well known.

Proposition 2.1 ([2]). *The following are equivalent.*

(1) $V(\lambda) \cong V(\mu)$ *as G-representations.*

(2) $\widetilde{V}(\lambda) \cong \widetilde{V}(\mu) \otimes \mathbb{C}_{\det^d}$ *as $\widetilde{G}$-representations for some integer d. Here* $\det : \widetilde{G} \to \mathbb{C}$ *is the determinant homomorphism and $\mathbb{C}_{\det^d}$ is the 1-dimensional representation defined by* $\det^d$.

(3) $\lambda - \mu = d(1, 1, 1)$ *for some integer d.*

Let $V = V(\lambda)$ be an irreducible G-representation. Then T-fixed space V^T is a W-representation, called the *zero weight representation* induced by V. On the other hand, $\widetilde{V}(\lambda)^T$ is a representation of $N_{\widetilde{G}}(T)$ $(= N_{\widetilde{G}}(\widetilde{T}))$ and by restricting to W_1, we may regard $\widetilde{V}(\lambda)^T$ as a W_1-representation. Since $V^T = \widetilde{V}(\widetilde{\lambda})^T$ as vector spaces, we see

Proposition 2.2 ([1]). $V^T = \widetilde{V}(\lambda)^T \neq 0$ *if and only if* $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 \equiv 0 \pmod 3$.

Let R_0 be the set of isomorphism classes of irreducible G-representations V with $V^T \neq 0$ and put the set

$$\Lambda_0 = \{\lambda \in \mathbb{Z}^3 \mid \lambda_1 \geq \lambda_2 \geq \lambda_3,\ \lambda_1 + \lambda_2 + \lambda_3 = 0\} \subset \Lambda.$$

Proposition 2.3. *R_0 is parametrized by Λ_0.*

Proof. We define a map $I : \Lambda_0 \to R_0$ by $I(\lambda) = V(\lambda)$.

Injectivity: If $V(\lambda) \cong V(\mu)$, then $\lambda - \mu = d(1,1,1)$ for some integer d. Since $|\lambda - \mu| = 0$, it follows that $d = 0$. Hence $\lambda = \mu$.

Surjectivity: Take any G-representation $V = \mathrm{res}_G \widetilde{V}(\widetilde{\lambda})$ with $V^T \neq 0$ and $\widetilde{\lambda} \in \Lambda$. Then, by Proposition 2.2, $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 + \widetilde{\lambda}_3 = 3d$ for some integer d. Put $\lambda_i = \widetilde{\lambda}_i - d$. Then $\lambda_1 + \lambda_2 + \lambda_3 = 0$ and we have $\mathrm{res}_G \widetilde{V}(\lambda) \cong V$ by Proposition 2.1. This implies that $I(\lambda) = V$. $\square$

For a nonnegative integer d, set

$$\Lambda(d) = \{(3d - k, k, 0) \mid 0 \leq k \leq [3d/2]\} \quad \text{and} \quad \Lambda_1 = \bigcup_{d \geq 0} \Lambda(d) \subset \Lambda,$$

where $[x]$ denotes the largest integer not exceeding x. We see the following by Proposition 2.1, see also [1].

Proposition 2.4. *For $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda_0$, define $\widetilde{\lambda} = (\lambda_1 - \lambda_3, \lambda_2 - \lambda_3, 0) \in \Lambda(d) \subset \Lambda_1$, where $d = -\lambda_3$.*

(1) *The correspondence $\lambda \mapsto \widetilde{\lambda}$ gives a bijection between Λ_0 and Λ_1.*

(2) *The W_1-representation $\widetilde{V}(\widetilde{\lambda})^T$ is regarded as a W-representation via ι. Then there is an isomorphism*

$$V(\lambda)^T \cong \widetilde{V}(\widetilde{\lambda})^T \otimes \mathbb{C}_{\mathrm{sgn}^d} = \begin{cases} \widetilde{V}(\widetilde{\lambda})^T & d : \textit{even} \\ \widetilde{V}(\widetilde{\lambda})^T \otimes \mathbb{C}_{\mathrm{sgn}} & d : \textit{odd}, \end{cases}$$

where $\mathrm{sgn} : W \to \mathbb{C}$ *is the sign representation.*

3. The W-fixed space of the zero weight representation

In this section, set $N = N_G(T)$ and $W = N/T$. Let $\widetilde{V} = \widetilde{V}(\lambda)$ and $V = V(\lambda) = \mathrm{res}_G \widetilde{V}(\lambda)$ for $\lambda \in \Lambda_1$. We shall investigate the W-fixed space $(V^T)^W = V^N$ of the W-representation V^T by the method of [1]; we call this method the Ariki-Matsuzawa-Terada algorithm (AMT-algorithm for short). In the case of $\widetilde{G}$, the AMT-algorithm gives the irreducible decomposition of $\widetilde{V}^T$ as W_1-representations, which leads to the irreducible decomposition of V^T as W-representations by Proposition 2.4.

The AMT-algorithm consists of three steps. The first step is to describe $\widetilde{V}$ as a linear combination of the symmetric tensor representations $S^k = S^k(\mathbb{C}^3_{\mathrm{nat}})$, where $\mathbb{C}^3_{\mathrm{nat}}$ is the natural representation of $\widetilde{G}$ on $\mathbb{C}^3$. According to [7, 1], we have

Proposition 3.1. *Let $\lambda = (\lambda_1, \lambda_2, 0) \in \Lambda_1$. Then*

$$\widetilde{V}(\lambda) = S^{\lambda_1} \otimes S^{\lambda_2} - S^{\lambda_1+1} \otimes S^{\lambda_2-1}$$

in the representation ring $R(\widetilde{G})$. Here if $k < 0$, then we set $S^k = 0$.

The second step is to decompose $(S^{\lambda_1} \otimes S^{\lambda_2})^T$ into a direct sum of permutation representations as W_1-representations, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 = 0$ and $\lambda_1 + \lambda_2 = 3d$. A basis of $S^{\lambda_1} \otimes S^{\lambda_2}$ as a vector space is given by

$$\mathcal{B} = \left\{ e^{\alpha_1} \otimes e^{\alpha_2} := (e_1^{\alpha_{11}} e_2^{\alpha_{12}} e_3^{\alpha_{13}}) \otimes (e_1^{\alpha_{21}} e_2^{\alpha_{22}} e_3^{\alpha_{23}}) \,\middle|\, \sum_{j=1}^{3} \alpha_{ij} = \lambda_i \ (i = 1, 2) \right\}.$$

The action of $t \in T$ on $e^{\alpha_1} \otimes e^{\alpha_2}$ is given by

$$t \cdot (e^{\alpha_1} \otimes e^{\alpha_2}) = (t_1^{\alpha_{11}+\alpha_{21}} t_2^{\alpha_{12}+\alpha_{22}} t_3^{\alpha_{13}+\alpha_{23}}) e^{\alpha_1} \otimes e^{\alpha_2}.$$

Thus we obtain that $e^{\alpha_1} \otimes e^{\alpha_2} \in (S^{\lambda_1} \otimes S^{\lambda_2})^T$ if and only if $\alpha_{1j} + \alpha_{2j} = d$ $(j = 1, 2, 3)$. Therefore a basis of $(S^{\lambda_1} \otimes S^{\lambda_2})^T$ as a vector space is given by

$$\mathcal{B}_0 = \left\{ e^{\alpha_1} \otimes e^{\alpha_2} \in \mathcal{B} \,\middle|\, \alpha_{1j} + \alpha_{2j} = d \ (j = 1, 2, 3) \right\}.$$

The W_1-action on $\mathcal{B}_0$ is given by permutations on $\{e_i\}$, or equivalently by column permutations on the 2×3 matrices (α_{ij}). Considering the lexicographical order on columns of (α_{ij}), we take matrices with nonincreasing columns as representatives of the orbit set $\mathcal{B}_0/W_1$, and denote by $\overline{\mathcal{B}_0}$ the set of such representatives. Furthermore, matrices $(\alpha_{ij}) \in \overline{\mathcal{B}_0}$ correspond bijectively to integer sequences $(c_0, \ldots, c_d)$ such that

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \sum_{i=0}^{d} c_i \begin{pmatrix} d-i \\ i \end{pmatrix}, \qquad 0 \leq c_i \leq 3, \quad \sum_{i=0}^{d} c_i = 3.$$

Let $\mathcal{C}_\lambda$ be the set of such integer sequences $(c_0, \ldots, c_d)$. There are three types of $(c_i) \in \mathcal{C}_\lambda$: (1) $c_i = 3$ for some i and the others are 0, (2) $c_i = 2$ and $c_j = 1$ for some $i \neq j$ and the others are 0, (3) $c_i = c_j = c_k = 1$ for distinct i, j, k and the others are 0. Let $\mathcal{L}_\lambda$ be the set of $(c_i) \in \mathcal{C}_\lambda$ with type (1), $\mathcal{M}_\lambda$ the set of $(c_i) \in \mathcal{C}_\lambda$ with type (2) and $\mathcal{N}_\lambda$ the set of $(c_i) \in \mathcal{C}_\lambda$ with type (3). Set $c_\lambda = \#\mathcal{C}_\lambda$, $l_\lambda = \#\mathcal{L}_\lambda$, $m_\lambda = \#\mathcal{M}_\lambda$, $n_\lambda = \#\mathcal{N}_\lambda$. Clearly $c_\lambda = l_\lambda + m_\lambda + n_\lambda$. Note also that if $d \equiv 0 \pmod 3$, then $c_{d/3} = 3$

and $l_\lambda = 1$, and $l_\lambda = 0$ otherwise. Thus, according to [1], we obtain the following decomposition into permutation representations:

Proposition 3.2 ([1]). $(S^{\lambda_1} \otimes S^{\lambda_2})^T \cong l_\lambda \mathbb{C} \oplus m_\lambda \mathbb{C}[S_3/S_2] \oplus n_\lambda \mathbb{C}[S_3]$.

The third step is the irreducible decomposition of a permutation representation. As is well-known, S_3 has three irreducible representations: $\mathbb{C}$, $\mathbb{C}_{\mathrm{sgn}}$ and one 2-dimensional representation, say U. The following decompositions are obtained from representation theory.

Proposition 3.3. (1) $\mathbb{C}[S_3/S_2] \cong \mathbb{C} \oplus U$.
(2) $\mathbb{C}[S_3] \cong \mathbb{C} \oplus \mathbb{C}_{\mathrm{sgn}} \oplus 2U$.

Combining these propositions, we obtain

Corollary 3.4.

$$(S^{\lambda_1} \otimes S^{\lambda_2})^T \cong c_\lambda \mathbb{C} \oplus n_\lambda \mathbb{C}_{\mathrm{sgn}} \oplus (m_\lambda + 2n_\lambda) U.$$

Any $\lambda \in \Lambda(d)$ is described as $\lambda = (3d - k, k, 0)$ for some $0 \leq k \leq [3d/2]$.

Definition. We set, for $\lambda = (3d - k, k, 0)$,

$$\widetilde{V}(k, d) = \widetilde{V}(\lambda)$$
$$V(k, d) = \mathrm{res}_G \widetilde{V}(k, d).$$

By Proposition 2.4, we have

$$\dim V(k, d)^N = \dim(V(k, d)^T)^W = \dim(\widetilde{V}(k, d)^T \otimes \mathbb{C}_{\mathrm{sgn}^d})^W.$$

Set $c(k, d) = c_\lambda$, $l(k, d) = l_\lambda$, $m(k, d) = m_\lambda$ and $n(k, d) = n_\lambda$. By Proposition 3.1, we have $\widetilde{V}(k, d) = S^{3d-k} \otimes S^k - S^{3d-k+1} \otimes S^{k-1}$. Since $\mathbb{C}_{\mathrm{sgn}} \otimes \mathbb{C}_{\mathrm{sgn}} = \mathbb{C}$ and $U \otimes \mathbb{C}_{\mathrm{sgn}} = U$, we obtain the following result by Proposition 2.4 and Corollary 3.4,

Proposition 3.5.

$$\dim V(k, d)^N = \begin{cases} c(k, d) - c(k - 1, d) & d : \textit{even} \\ n(k, d) - n(k - 1, d) & d : \textit{odd} \end{cases}$$

for $0 \leq k \leq [3d/2]$.

For example, the values of $r(k, d) := \dim V(k, d)^N / \dim V(k, d)$ for $1 \leq d \leq 4$ and $0 \leq k \leq [3d/2]$ are given in Tables below. Recall

$$\dim V(k, d) = (2k - 3d - 1)(k - 3d - 2)(k + 1)/2$$

by the dimension formula.

k	$\dim V(k,d)^N$	$\dim V(k,d)$	$r(k,d)$
0	0	10	0
1	0	8	0

TABLE 1. $d = 1$

k	$\dim V(k,d)^N$	$\dim V(k,d)$	$r(k,d)$
0	1	28	1/28
1	0	35	0
2	1	27	1/27
3	0	10	0

TABLE 2. $d = 2$

k	$\dim V(k,d)^N$	$\dim V(k,d)$	$r(k,d)$
0	0	55	0
1	0	80	0
2	0	81	0
3	1	64	1/64
4	0	35	0

TABLE 3. $d = 3$

k	$\dim V(k,d)^N$	$\dim V(k,d)$	$r(k,d)$
0	1	91	1/91
1	0	143	0
2	1	162	1/162
3	1	154	1/154
4	1	125	1/125
5	0	81	0
6	1	28	1/28

TABLE 4. $d = 4$

We next note the following.

Lemma 3.6. *Let $k' = 3d - 2k$ and $d' = 2d - k$, where $0 \le k \le [3d/2]$.*

(1) *$0 \le k' \le [3d'/2]$, and if $k > d$, then $k' < d'$.*

(2) *$\overline{V}(k,d) \cong V(k',d')$, where $\overline{V}(k,d)$ is the complex conjugate representation of $V(k,d)$.*

(3) *$\dim V(k,d) = \dim V(k',d')$ and $\dim V(k,d)^N = \dim V(k',d')^N$.*

Proof. (1) is straightforward.

(2) Since $V(k,d) \cong V(\lambda)$ for $\lambda = (2d-k, k-d, -d) \in \Lambda_0$, the highest weight of $\overline{V}(k,d)$ is given by $\lambda^* = (d, d-k, k-2d) \in \Lambda_0$. Hence we see that $\overline{V}(k,d) \cong V(\mu)$ for $\mu = (3d'-k', k', 0) = (3d-k, 3d-2k, 0) \in \Lambda_1$.

(3) This follows from (2). $\square$

As an example, we see that $r(6,4) = r(0,2)$ as in Tables 2 and 4. Thus, by Lemma 3.6, we may discuss the estimation only in the range of $0 \le k \le d$.

The next theorem and its corollary give a complete computation of $\dim V^N$ for all irreducible G-representations V.

Theorem 3.7. *Let $V(k,d)$ be as before, where $d \ge 0$ and $0 \le k \le d$. Then*

$$\dim V(k,d)^N = \begin{cases} \left[\dfrac{k}{2}\right] - \left[\dfrac{k-1}{3}\right] & d : \textit{even} \\[2ex] \left[\dfrac{k-1}{2}\right] - \left[\dfrac{k-1}{3}\right] & d : \textit{odd} \end{cases}$$

Proof. If $d = 0$, then $k = 0$, and then $V(k,d)$ is trivial, and so $\dim V(k,d)^N = 1$. In the case of $d = 1$, it is clear from Table 1. Fix $d \ge 2$ and set $\mathcal{C}(k) = \mathcal{C}_\lambda$ for $\lambda = (3d-k, k, 0)$, $0 \le k \le d$. We first compute $c(k,d)$. If $k = 0$, then $\mathcal{C}(0)$ consists of only one element $(3, 0, \ldots, 0)$; hence $c(0,d) = 1$. In the following, assume $k \ge 1$. Define

$$\mathrm{imax}(c_i) := \max\{i \mid c_i \ne 0\}$$

for $(c_i) \in \mathcal{C}(k)$. Let $t = \mathrm{imax}(c_i)$ for $(c_i) \in \mathcal{C}(k-1)$. Since

$$(\ \ldots,\ \overset{t}{c_t - 1},\ \overset{t+1}{1},\ 0,\ \ldots,\ 0\ ) \in \mathcal{C}(k),$$

a function $S : \mathcal{C}(k-1) \to \mathcal{C}(k)$ can be defined by

$$S :\ (\ \ldots,\ \overset{t}{c_t},\ \overset{t+1}{0},\ \ldots,\ 0\ ) \mapsto (\ \ldots,\ \overset{t}{c_t - 1},\ \overset{t+1}{1},\ 0,\ \ldots,\ 0\ ).$$

It is easily seen that S is injective. Since $S(\mathcal{C}(k-1))$ consists of $(c_i) \in \mathcal{C}(k)$ with $\mathrm{imax}(c_i) = 1$, it follows that $\mathcal{C}(k) \setminus S(\mathcal{C}(k-1))$ consists of (c_i) such that $c_s = 1$, $c_t = 2$ for some s, t ($s < t$ and $s + 2t = k$) or $c_t = 3$ for some t ($3t = k$). Thus this set corresponds bijectively to

$$\mathcal{D}(k) := \{(s,t) \in \mathbb{Z}^2 \mid s + 2t = k,\ 0 \le s \le t \le d\}.$$

As easily seen, $(s,t) \in \mathcal{D}(k)$ are corresponding bijectively to integers t such that $k/3 \le t \le k/2$, and this inequality is equivalent to $[(k+2)/3] \le t \le [k/2]$. Consequently, when d is even, we obtain

$$\dim V(k,d)^N = c(k,d) - c(k-1,d) = \left[\frac{k}{2}\right] - \left[\frac{k-1}{3}\right].$$

Similarly, we set $\mathcal{N}(k) := \mathcal{N}_\lambda$ for $\lambda = (3d-k, k, 0)$. Clearly if $d = 1$, then $\mathcal{C}(k) = \emptyset$ for $k = 0, 1$, and so $\dim V(k,d)^N = 0$. In the following, assume $d \geq 3$. For any $(c_i) \in \mathcal{N}(k-1)$, let $u = \mathrm{imax}(c_i)$. Since

$$(\ \ldots,\ \overset{u}{0},\ \overset{u+1}{1},\ 0,\ \ldots,\ 0\ ) \in \mathcal{N}(k),$$

there is an injective function $T : \mathcal{N}(k-1) \to \mathcal{N}(k)$ defined by

$$T:\ (\ \ldots,\ \overset{u}{1},\ \overset{u+1}{0},\ \ldots,\ 0\ ) \mapsto (\ \ldots,\ \overset{u}{0},\ \overset{u+1}{1},\ 0,\ \ldots,\ 0\ ).$$

Hence $\mathcal{N}(k) \setminus T(\mathcal{N}(k-1))$ consists of elements

$$(c_i) = (\ \ldots,\ \overset{s}{1},\ \ldots,\ \overset{t}{1},\ \overset{t+1}{1},\ \ldots\ )$$

with $s + 2t + 1 = k$ and $0 \leq s < t < d$. This set corresponds bijectively to

$$\mathcal{E}(k) := \{(s,t) \in \mathbb{Z}^2 \mid s + 2t + 1 = k,\ 0 \leq s < t < d\},$$

and the elements of $\mathcal{E}(k)$ correspond bijectively to integers t such that $(k-1)/3 < t \leq (k-1)/2$ or equivalently $[(k+2)/3] \leq t \leq [(k-1)/2]$. Consequently, when d is odd, we obtain

$$\dim V(k,d)^N = n(k,d) - n(k-1,d) = \left[\frac{k-1}{2}\right] - \left[\frac{k-1}{3}\right].$$

□

Corollary 3.8. *If $d < k \leq [3d/2]$, then*

$$\dim V(k,d)^N = \begin{cases} \left[\frac{d}{2}\right] - \left[\frac{k-1}{3}\right] & k : \textit{even} \\ \left[\frac{d-1}{2}\right] - \left[\frac{k-1}{3}\right] & k : \textit{odd} \end{cases}$$

Proof. This follows from Lemma 3.6. □

4. Proof of Theorem 1.1

In this section, $\widetilde{G} = \mathrm{U}(3)$, $G = \mathrm{SU}(3)$, $\widetilde{N} = N_{\widetilde{G}}(\widetilde{T}) = \widetilde{T} \rtimes W_1$ and $N = N_G(T)$ as before. Since N is solvable, we have

Lemma 4.1. $c_N = 1$.

We define

$$d_{(G,N)} = \sup_V \{\dim V^N / \dim V\},$$

where V is taken over all nontrivial irreducible G-representations. The following proposition can be proved by a similar argument of [3].

Proposition 4.2. $c_G \geq 1 - d_{(G,N)}$.

Thus Theorem 1.1 follows from the next result.

Theorem 4.3. $d_{(G,N)} = 1/27$.

The rest of this section is devoted to the proof of Theorem 4.3. First we remark the following.

Lemma 4.4. $\dim V(k,d)^N \leq \dfrac{k+6}{6}$ *for any* $d \geq 1$ *and* $0 \leq k \leq d$.

Proof. It follows from Theorem 3.7 that

$$\dim V(k,d)^N \leq \left[\frac{k}{2}\right] - \left[\frac{k-1}{3}\right].$$

Furthermore a case-by-case consideration shows

$$\left[\frac{k}{2}\right] - \left[\frac{k-1}{3}\right] \leq \frac{k+6}{6}.$$

□

Next we show the following.

Lemma 4.5. *If* $d \geq 3$*, then*

$$r(k,d) = \dim V(k,d)^N / \dim V(k,d) \leq 1/28$$

for $0 \leq k \leq d$.

Proof. By Lemma 4.4 and

$$\dim V(k,d) = (2k-3d-1)(k-3d-2)(k+1)/2,$$

it suffices to show that

$$\frac{(k+6)/6}{(2k-3d-1)(k-3d-2)(k+1)/2} \leq 1/28,$$

or equivalently,

$$6k^3 - (9d+27)k^2 + (27d^2-37)k + 27d^2 + 27d - 162 \geq 0$$

for $0 \le k \le d$. We consider the function

$$f(x) = 6x^3 - (9d+27)x^2 + (27d^2 - 37)x + 27d^2 + 27d - 162.$$

Then

$$\frac{d^2 f}{dx^2}(x) = -18(1 + 3d - 2x) < 0$$

for $0 \le x \le d$ and so f is upper convex. Therefore it follows that

$$f(k) \ge \min\{f(0), f(d)\}.$$

It is easy to see that

$$f(0) = 27(d^2 + d - 6) \ge 0 \quad \text{if} \ \ d \ge 2$$

and

$$f(d) = 2(3d^3 + 9d^2 - 5d - 81) > 0 \quad \text{if} \ \ d \ge 3.$$

Consequently it turns out that $f(k) \ge 0$ if $d \ge 3$. □

Proof of Theorem 4.3. By a similar argument of [3], it suffices to consider unitary G-representations in order to compute $d_{(G,N)}$. This implies

$$d_{(G,T)} = \sup\{r(k,d) \mid d \ge 1,\ 0 \le k \le [3d/2]\}.$$

By Lemma 3.6, we may assume that $0 \le k \le d$. Thus

$$d_{(G,T)} = \sup\{r(k,d) \mid d \ge 1,\ 0 \le k \le d\}.$$

By Lemma 4.5,

$$\sup\{r(k,d) \mid d \ge 3,\ 0 \le k \le d\} \le 1/28.$$

Looking at Tables 1 and 2, we see that the maximum of $r(k,d)$ is $r(2,2) = 1/27$ among the cases $d = 1$, 2. Eventually we obtain $d_{(G,T)} = 1/27$. □

Remark. Since $\overline{V}(2,2) \cong V(2,2)$, $V(2,2)$ is only one irreducible G-representation attaining the maximum $1/27$.

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