Representations with the Borsuk-Ulam property

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Abstract. In the previous study, we defined the equivariant level and colevel of representations. In this paper, we introduce the notion of Borsuk-Ulam property of representations using these invariants. Furthermore, we show some fundamental properties and provide illustrative examples of representations with the Borsuk-Ulam property.

1. **Introduction**

Let V be an (orthogonal) G-representation, where G is a compact Lie group. In [4, 5], we have determined the compact Lie groups with the Borsuk-Ulam property. In particular, the Borsuk-Ulam theorem does not hold unless G is an elementary abelian p-group or a torus. However, in a certain class of representations, the Borsuk-Ulam theorem still holds. For example, if V and W are free G-representations, i.e., G acts freely on $S(V)$ and $S(W)$, then the Borsuk-Ulam theorem holds.

In [6], we introduced the equivariant level $l_G(X)$ and colevel $cl_G(X)$ of a fixed-point-free G-space X. For a fixed-point-free G-representation sphere $S(V)$, we set $l_G(V) = l_G(S(V))$ and $r_G(V) = cl_G(S(V))$.

Definition. (1) We say that V has the *left Borsuk-Ulam property* (LBUP) if $l_G(V) = \dim V$.

(2) We say that V has the *right Borsuk-Ulam property* (RBUP) if $r_G(V) = \dim V$.

(3) We say that V has the *Borsuk-Ulam property* (BUP) if $l_G(V) = r_G(V) = \dim V$, i.e., V has LBUP and RBUP.

We say that V is fixed-point-free if $S(V)$ is fixed-point-free, i.e., $V^G = 0$, and also that V is free if G acts freely on $S(V)$. The Borsuk-Ulam theorem for G-maps is equivalent to that any fixed-point-free G-representation has BUP. Therefore, if G

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is an elementary abelian p-group C_p^k or a k-torus T^k, then any fixed-point-free G-representation has BUP.

In this paper, we show some fundamental properties of $l_G(V)$ and $r_G(V)$ and provide some examples of G-representations with the left or right Borsuk-Ulam property.

2. Fundamental facts of the Borsuk-Ulam property

Unless otherwise stated below, we assume that G-representations are orthogonal and fixed-point-free, and a subgroup of G means a closed subgroup. Furthermore all maps between spaces are assumed to be continuous.

We first recall the definition of $l_G(S(V))$ and $cl_G(S(V))$ from [6]. Let $L_G(S(V))$ be the set of G-representations W such that there exists a G-map $f : S(V) \to S(W)$, and $CL_G(S(V))$ the set of G-representations U such that there exists a G-map $g : S(U) \to S(V)$.

Definition.
(1) $l_G(V) = l_G(S(V)) := \inf\{\dim W \mid W \in L_G(S(V))\}$.
(2) $r_G(V) = cl_G(S(V)) := \sup\{\dim U \mid U \in CL_G(S(V))\}$.

As a convention, we set $l_G(V) = r_G(V) = 0$ when $V = 0$. Clearly

$$0 \le l_G(V) \le \dim V \le r_G(V) \le \infty.$$

Definition. We say that a G-map $f : S(V) \to S(U)$ realizes $l_G(V)$ if $l_G(V) = \dim U$. In this case we also say that the G-representation U realizes $l_G(V)$. Similarly, we say that a G-map $f : S(U) \to S(V)$ or U realizes $r_G(V)$ if $r_G(V) = \dim U$.

We first note the following.

Proposition 2.1. *If U realizes $l_G(V)$ [resp. $r_G(V)$], then U has LBUP [resp. RBUP].*

Proof. Let $f : S(V) \to S(U)$ be a G-map realizing $l_G(V)$. If U does not have LBUP, then there is a G-map $g : S(U) \to S(W)$ with $\dim U > \dim W$. Then $g \circ f : S(V) \to S(W)$ implies $l_G(V) > \dim W$. This is a contradiction. The case of RBUP is similar. □

Proposition 2.2. *If V has LBUP [resp. RBUP], then any sub-representation U of V has LBUP [resp. RBUP].*

Proof. Let $V = U \oplus U_1$. If U does not have LBUP, there exists a G-map $f : S(U) \to S(W)$, $\dim U > \dim W$. Then we have a G-map $f * id : S(V) = S(U \oplus U_1) \to S(W \oplus U_1)$. This contradicts that V has LBUP. The case of RBUP is similar. □

Corollary 2.3. *If U has LBUP [resp. RBUP], then U^H has LBUP [resp. RBUP] for any normal subgroup H of G.* □

Proof. Since U^H is a sub-representation of U, the result follows. □

Proposition 2.4. *Let $Q = G/H$ be a quotient group of G. Then $l_G(\mathrm{Inf}_Q^G V) = l_Q(V)$. In particular, a Q-representation V has LBUP if and only if the inflation $\mathrm{Inf}_Q^G V$ has LBUP.*

Proof. This is shown by Proposition 2.2 of [6]. □

Remark. The equality $r_G(\mathrm{Inf}_Q^G V) = r_Q(V)$ does not hold in general. Hence $\mathrm{Inf}_Q^G V$ need not have RBUP even if V has RBUP.

Definition. Let V and W be G-representations. We say that V and W are *dimensionally equivalent* if $\dim V^H = \dim W^H$ for any subgroup of G.

The following propositions hold.

Proposition 2.5. (1) *If $f : S(V) \to S(W)$ is a G-map, then $l_G(V) \le l_G(W)$ and $r_G(V) \le r_G(W)$.*

(2) *If V and W are dimensionally equivalent, then $l_G(V) = l_G(W)$ and $r_G(V) = r_G(W)$. In particular, V has LBUP [resp. RBUP] if and only if W has LBUP [resp. RBUP].*

Proof. (1) This is clear by the definition.

(2) By equivariant obstruction theory [2], there are G-maps $\alpha : S(V) \to S(W)$ and $\beta : S(W) \to S(V)$. Therefore $l_G(V) = l_G(W)$ and $r_G(V) = r_G(W)$ by (1). □

Proposition 2.6. *Let V be a fixed-point-free G-representation and H a subgroup of G. If $\mathrm{res}_H V$ is free H-representation, then V has RBUP as a G-representation.*

Proof. If $f : S(U) \to S(V)$ is a G-map, then $\mathrm{res}_H f$ is an H-map between free H-representation spheres. By Borsuk-Ulam theorem, $\dim U \le \dim V$. Thus V has RBUP as a G-representation. □

Remark. A free G-representation need not have LBUP. In fact, such a counterexample is provided by Theorem 4.2 (2) in section 4.

3. Low dimensional cases

In this section, we discuss the Borsuk-Ulam property of G-representations of low dimensions.

When $\dim V = 0$, V has BUP by convention. When $\dim V = 1$, $S(V)$ consists of two points with non-trivial action. Therefore there are no G-maps from $S(U)$ with $\dim U \geq 2$ since $S(U)$ is connected. Thus we have

Proposition 3.1. *If* $\dim V = 1$*, then* V *has BUP.* □

Next, we consider the case of $\dim V = 2$. We begin with the following lemma.

Lemma 3.2. *Let* C_n *be a (non-trivial) cyclic group of order* n *and* U, V *(fixed-point-free) 2-dimensional* C_n*-representations. Then the degree of any* C_n*-map* $f : S(U) \to S(V)$ *is non-zero.*

Proof. A 2-dimensional C_n-representation $T_k\,(= \mathbb{C})$, $k \in \mathbb{Z}/n$, is given by $a \cdot z = \xi_n^k x$, $z \in T_k$, where $\xi_n = \exp(2\pi\sqrt{-1}/n)$. When $k \neq 0$ and $k \neq n/2$ if n is even, T_k is irreducible and $T_k \cong T_{-k}$. Note that $T_0 = 2\mathbb{R}$ and $T_{n/2} \cong 2\mathbb{R}_\varepsilon$, where $\mathbb{R}_\varepsilon$ is given by $a \cdot x = -x$, $x \in \mathbb{R}_\varepsilon$.

If necessary, considering $U^{\mathrm{Ker}\,U}$, one may suppose that U is faithful. Then U is a free C_n-representation. One can set $U = T_m$ and $V = T_l$ ($0 < m < n$, $0 < l < n$) and $(m, n) = 1$. Let k be an integer such that $km \equiv 1 \bmod n$. By a result of [2], one sees $\deg f \equiv kl \not\equiv 0 \bmod n$. In particular, $\deg f \neq 0$. □

We shall show

Proposition 3.3. *If* $\dim V = 2$*, then* V *has BUP.*

Proof. By Proposition 3.1, V has LBUP. In fact, if there exists a G-map $f : S(V) \to S(W)$, $\dim V > \dim W$, then $\dim W = 1$, however, this contradicts that W has BUP. We shall show that V has RBUP. If V does not have RBUP, then there is a G-map $f : S(U) \to S(V)$ with $\dim U \geq 3$. Let $K = \mathrm{Ker}\,V$. Since $\dim V = 2$, G/K is a non-trivial subgroup of $\mathrm{O}(2)$ and hence G/K is isomorphic to a cyclic group C_m ($m \geq 2$), a dihedral group D_m ($m \geq 2$), S^1 or $\mathrm{O}(2)$. Thus V is a 2-dimensional irreducible representation or a direct sum of two non-trivial 1-dimensional representations. The latter case happens only when $G/K = C_2$ or $D_2 = C_2 \times C_2$. Since V^K is faithful, in any case, there exists a subgroup L including K such that $L/K \cong C_p$ for some prime p and L/K acts freely on $V^K = V$. Let $\bar{a}$ be a generator of L/K. Take $a \in G$

such that $\pi(a) = \bar{a}$, where $\pi : G \to G/K$ is the projection. Set $H = \overline{\langle a \rangle} \leq G$, which is isomorphic to $T^k \times C_l$ for some k, l. Since $V^H = 0$, one can take a finite cyclic subgroup $C \leq H$ such that $V^C = 0$ and so $U^C = 0$. Restricting to C, we obtain a C-map $g = \mathrm{res}_C f : S(U) \to S(V)$. Since U decomposes into a direct sum of 1- or 2-dimensional irreducible representations as C-representations, U has a 2-dimensional sub-representation U_1. We obtain a C-map $g_{|S(U_1)} : S(U_1) \to S(V)$. By Lemma 3.2, $\deg g_{|S(U_1)} \neq 0$. On the other hand, $S(U_1) \cong S^1$ bounds a 2-disk in $S(U)$, hence $\deg g_{|S(U_1)} = 0$. This is a contradiction. Therefore U has RBUP. □

Corollary 3.4. *If* (*i*) $\dim V = 3$, *or* (*ii*) *if* $\dim V = 4$ *and* G/G_0 *is of odd order, then* V *has LBUP.*

Proof. Note that if G/G_0 is of odd order, then a fixed-point-free representation is even dimensional. If V does not have LBUP, then there is a G-map $f : S(V) \to S(W)$, $\dim W = 2$, however this is impossible by Proposition 3.3. □

4. The Borsuk-Ulam property of C_{pq}-representations

In this section, we consider the case of $G = C_{pq}$, where p, q are primes. Let a be a generator of G. The 2-dimensional G-representation T_k $(= \mathbb{C})$, $k \in \mathbb{Z}/pq$, is given by $a \cdot z = \xi_{pq}^k x$, $z \in T_k$, where $\xi_{pq} = \exp(\frac{2\pi\sqrt{-1}}{pq})$. When p and q are odd primes, T_k is irreducible for $k \neq 0$. If p is an odd prime and $q = 2$, then $T_0 = 2\mathbb{R}$ and $T_p \cong 2\mathbb{R}_\varepsilon$, where $\mathbb{R}_\varepsilon$ is given by $a \cdot x = -x$, $x \in \mathbb{R}_\varepsilon$, and the others are irreducible.

Note the following.

Lemma 4.1. *Let* $G = C_{pq}$, *where* p, q *are distinct primes.*

(1) *If* $p > q \geq 3$, *then a* G*-representation* V *is dimensionally equivalent to* $a_1T_1 \oplus a_pT_p \oplus a_qT_q$ *for some non-negative integers* a_i.
(2) *If* $p > q = 2$, *then a* G*-representation* V *is dimensionally equivalent to* $a_1T_1 \oplus a_2T_2 \oplus a_p\mathbb{R}_\varepsilon$ *for some non-negative integers* a_i.
(3) *If* $p = q \geq 3$, *then a* G*-representation* V *is dimensionally equivalent to* $a_1T_1 \oplus a_pT_p$ *for some non-negative integers* a_i.
(4) *If* $p = q = 2$, *then a* G*-representation* V *is dimensionally equivalent to* $a_1T_1 \oplus a_2\mathbb{R}_\varepsilon$ *for some non-negative integers* a_i.

We first consider the case where p, q are distinct odd primes. We may assume that

$$V = a_1T_1 \oplus a_pT_p \oplus a_qT_q$$

for some non-negative integers a_1, a_p, a_q by Lemma 4.1.

Theorem 4.2. *Assume $p > q \geq 3$ and coefficients a_i are positive integers.*

(1) *$V = a_pT_p$ and $V = a_qT_q$ have BUP.*
(2) *$V = a_1T_1$ has RBUP. If $a_1 \leq 2$, then V has BUP. If $a_1 \geq 3$, then V does not have LBUP.*
(3) *$V = a_pT_p \oplus a_qT_q$ has LBUP and does not have RBUP.*
(4) *$V = a_1T_1 \oplus a_pT_p$ and $V = a_1T_1 \oplus a_qT_q$ have RBUP. If $a_1 = 1$, then V has BUP and if $a_1 \geq 2$, then V does not have LBUP.*
(5) *$V = a_1T_1 \oplus a_pT_p \oplus a_qT_q$ has neither LBUP nor RBUP.*

Proof. Essentially, this follows from Theorem 3.3 of [6]. We here give a sketch of the proof.

(1) This follows from the Borsuk-Ulam theorem and Proposition 2.6.

(2) This follows from Proposition 3.3 and the existence of a G-map $f : S(3T_1) \to S(T_p \oplus T_q)$.

(3) This follows from the Borsuk-Ulam theorem and and the existence of a G-map $f : S(T_1 \oplus a_pT_p \oplus a_qT_q) \to S(a_pT_p \oplus a_qT_q)$.

(4) This follows from Proposition 2.6 and the existence of a G-maps $f : S(V) \to S(a_pT_p \oplus T_q)$.

(5) This follows from the existence of G-maps $f : S(V) \to S(a_pT_p \oplus a_qT_q)$ and $g : S(T_1 \oplus V) \to S(V)$. □

Corollary 4.3. *Assume $p > q \geq 3$. Any G-representation with BUP is dimensionally equivalent to one of the following: a_pT_p, a_qT_q, T_1, $2T_1$, $T_1 \oplus a_pT_p$, $T_1 \oplus a_qT_q$.*

Next, we consider $G = C_{2p}$, where p is an odd prime. By Lemma 4.1, we may assume

$$V = a_1T_1 \oplus a_2T_2 \oplus a_p\mathbb{R}_\varepsilon.$$

We prepare the following.

Lemma 4.4. *Let k be a positive integer such that $2k \equiv 1 \bmod p$. The map $h : T_1 \oplus T_2 \to T_2 \oplus \mathbb{R}_\varepsilon$ defined by*

$$h(z, w) = (z^2 - w^{2k}, z^p\bar{w}^{kp} + \bar{z}^p w^{kp})$$

is a G-map and $h^{-1}(\{0\}) = \{0\}$. Thus h induces a G-map $f : S(T_1 \oplus T_2) \to S(T_2 \oplus \mathbb{R}_\varepsilon)$.

Proof. This h is given by a similar way as [1]. Let a be a generator of C_{2p}. Since $a(z,w) = (\xi_{2p}z, \xi_{2p}^2 w)$ and $4k \equiv 2 \bmod 2p$,

$$\begin{aligned} h(a(z,w)) &= (\xi_{2p}^2 z^2 - \xi_{2p}^{4k} w^{2k},\ \xi_{2p}^{p}\xi_{2p}^{-2kp} z^p \bar{w}^{kp} - \xi_{2p}^{-p}\xi_{2p}^{2kp} z^p \bar{w}^{2kp}) \\ &= (\xi_{2p}^2(z^2 - w^{2k}), -(z^p\bar{w}^{kp} + \bar{z}^p w^{kp})) \\ &= ah(z,w) \end{aligned}$$

Thus h is a G-map. If $h(z,w) = 0$, then $z = \pm w^k$ and $z^p\bar{w}^{kp} + \bar{z}^p w^{kp} = \pm 2|w|^{2kp} = 0$. Thus $w = z = 0$. □

Proposition 4.5. *For $V = T_1 \oplus T_1$ and $V = T_1 \oplus T_2$, it follows that $l_G(V) = 3$ and hence V does not have LBUP. On the other hand $T_2 \oplus \mathbb{R}_\varepsilon$ does not have RBUP.*

Proof. By Lemma 4.4, there exists a G-map $f : S(T_1 \oplus T_i) \to S(T_2 \oplus \mathbb{R}_\varepsilon)$, $i = 1, 2$. Thus $l_G(V) \le 3$. By Proposition 3.3, it follows that $l_G(V) = 3$. □

Corollary 4.6. *$V = a_1T_1 \oplus a_2T_2 \oplus a_p\mathbb{R}_\varepsilon$, $a_1 \ge 1$, $a_2 \ge 1$ and $a_p \ge 1$, has neither LBUP nor RBUP.*

Proposition 4.7. *Let $V = a_1T_1 \oplus a_p\mathbb{R}_\varepsilon$. If $a_1 = 1$, then V has BUP. If $a_1 \ge 2$, then V does not have LBUP.*

Proof. Since V is C_2-free, it follows form Proposition 2.6 that V has RBUP. If $a_1 \ge 2$, then by Proposition 4.5, V does not have LBUP. Suppose that $a_1 = 1$ and $f : S(V) \to S(W)$, $\dim W < \dim V$, is a G-map. Since $a_p \le \dim W(C_p)$, if follows that $W = a_p\mathbb{R}_\varepsilon$ or $(a_p + 1)\mathbb{R}_\varepsilon$. However, each case is impossible by the Borsuk-Ulam theorem. □

By the above facts, we obtain the following.

Theorem 4.8. *Assume that p is an odd prime and coefficients a_i are positive integers.*

(1) *$V = a_2T_2$ or $V = a_p\mathbb{R}_\varepsilon$ has BUP.*
(2) *$V = a_1T_1$ has RBUP. V has BUP if $a_1 = 1$ and does not have LBUP if $a_1 \ge 2$.*
(3) *$V = a_2T_2 \oplus a_p\mathbb{R}_\varepsilon$ has LBUP and does not have RBUP.*
(4) *$V = a_1T_1 \oplus a_2T_2$ has RBUP and does not have LBUP.*
(5) *$V = a_1T_1 \oplus a_p\mathbb{R}_\varepsilon$ has RBUP and has BUP if $a_p = 1$ and does not have LBUP if $a_p \ge 2$.*
(6) *$V = a_1T_1 \oplus a_2T_2 \oplus a_p\mathbb{R}_\varepsilon$ has neither LBUP nor RBUP.*

Corollary 4.9. *Let $G = C_{2p}$, where p is an odd prime. Any G-representation with BUP is dimensionally equivalent to one of the following: a_2T_2, $a_p\mathbb{R}_\varepsilon$, T_1, $T_1 \oplus a_p\mathbb{R}_\varepsilon$.*

Finally we consider the case of C_{p^2}. We may suppose that $V = a_1T_1 \oplus a_pT_p$ if p is an odd prime and $V = a_1T_1 \oplus a_2\mathbb{R}_\varepsilon$ if $p = 2$. The following fact is known, see for example [1].

Example 4.10. *Let $G = C_{p^2}$.*

(1) *If p is an odd prime, then there exists a G-map $f : S(4T_1) \to S(3T_p)$.*
(2) *If $p = 2$, then there exists a G-map $f : S(2T_1) \to S(3\mathbb{R}_\varepsilon)$.*

This example and Propositions 3.1 and 3.3 lead to the following result.

Proposition 4.11. *Let $G = C_{p^2}$.*

(1) *$V = a_1T_1$ has RBUP, and does not have LBUP if $a_1 \geq 4$.*
(2) *$V = T_1$ has BUP.*
(3) *$V = 2T_1$ has BUP if p is an odd prime and does not have LBUP if $p = 2$.*
(4) *When p is an odd prime, $V = a_1T_1 \oplus a_pT_p$ does not RBUP if $a_p \geq 3$ and does not have LBUP if $a_1 \geq 4$.*
(5) *When $p = 2$, $V = a_1T_1 \oplus a_2\mathbb{R}_\varepsilon$ does not have RBUP if $a_2 \geq 3$ and does not have LBUP if $a_1 \geq 2$.*

Corollary 4.12. *T_1, $2T_1$ (p:odd), T_p (p:odd), $\mathbb{R}_\varepsilon$ and $2\mathbb{R}_\varepsilon$ have BUP.*

Several unsolved cases are left. For example, consider the case of $V = 3T_1$. Since V is free G-representation, V has RBUP. On the other hand, we have $l_G(V) = 5$ if $p = 2$ by a result of [7], and $l_G(V) = 4$ or 6 by a result of [3]. Thus V does not have LBUP if $p = 2$. When p is an odd prime, this is an open problem.

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