On the zero weight ratios of exceptional compact Lie groups

Ikumitsu NAGASAKI [1)]

Abstract. The zero weight ratio d_G of a connected compact Lie group G with a maximal torus T is defined as the supremum of $r(V) = \dim V^T / \dim V$ for all nontrivial irreducible G-representations V. The zero weight ratio relates to a lower bound of the isovariant Borsuk-Ulam constant c_G such as $c_G \geq 1 - d_G$. In our previous research, we presented the value of d_G for any simple compact Lie group; however, we omitted the detailed proof in some cases. In this article, we provide the proof for exceptional compact Lie groups.

1. Introduction

Let G be a compact Lie group. Let V and W be G-representations and denote by $S(V)$ and $S(W)$ G-representation spheres, that is, the unit spheres of V, W. The isovariant Borsuk-Ulam constant c_G is defined as the supremum of constants $c \in \mathbb{R}$ such that the Borsuk-Ulam inequality

$$c(\dim V - \dim V^G) \leq \dim W - \dim W^G$$

holds whenever there exists a G-isovariant map $f : S(V) \to S(W)$. In [4], we have introduced the *zero weight ratio* d_G, which is defined as the supremum of

$$r(V) = \dim V^T / \dim V$$

for all nontrivial irreducible G-representations V, and we have proved that $c_G \geq 1 - d_G$.

The purpose of this article is to give the detailed proof of the following result.

Theorem 1.1 ([4])**.** *The zero weight ratios of exceptional compact Lie groups are given in Table 1. Furthermore, d_G is attained by $r(L(G))$ of the adjoint representation $L(G)$ of G.*

Note that d_G depends only on the type of G by Lemma 3.2 of [4].

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Type of G	E_6	E_7	E_8	F_4	G_2
d_G	$\frac{1}{13}$	$\frac{1}{19}$	$\frac{1}{31}$	$\frac{1}{13}$	$\frac{1}{7}$

TABLE 1. The values of d_G

2. Basic tools

In order to prove Theorem 1.1, we recall necessary notations and results from [4]. Throughout of this section, a representation is unitary and its dimension means the complex dimension. Note that d_G does not depend on whether representations are orthogonal or complex by Lemma 3.1 of [4].

Let G be a 1-connected compact simple Lie group of rank n and T a (fixed) maximal torus of G. The Weyl group is denoted by W. The 1-connected compact simple Lie groups (or their Lie algebras) are classified by their types and types E_l ($l = 6, 7, 8$), F_4 and G_2 are called *exceptional*. Types A_n, D_n, E_l, $l = 6, 7, 8$, are called *simply laced* and the other types B_n, C_n, F_4 and G_2 are called *nonsimply laced*.

Let R be the root system and R^+ the set of positive roots. The simple roots are denoted by $\alpha_1, \ldots, \alpha_n$. Let Λ be the weight lattice and Λ^+ the set of dominant weights. The fundamental weights ω_i are defined by

$$\langle \omega_i, \alpha_j \rangle := 2(\omega_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$$

for any i, j, where $(-,-)$ denote the standard (W-invariant) euclidean inner product. More precisely, we take $(-,-)$ as one on $\mathbb{R}^8$ for E_l, on $\mathbb{R}^4$ for F_4 and on $\mathbb{R}^3$ for G_2, see Appendix. The partial order $\leq$ on Λ is defined by

$$\mu \leq \lambda \iff \lambda - \mu = \sum_{i=1}^{n} k_i \alpha_i \text{ for some } k_i \in \mathbb{Z}_+.$$

Here $\mathbb{Z}_+$ is the set of nonnegative integers. Note that $\lambda \in \Lambda$ is a linear combination of ω_i with integer coefficients: $\lambda = \sum_i m_i \omega_i$, and $\lambda \in \Lambda^+$ if and only if $m_i \geq 0$ for every i.

The Weyl group W acts on R and the orbit set R/W consists of one orbit for simply laced types and two orbits for nonsimply laced types. We take a representative ν for E_l, $l = 6, 7, 8$, and representatives ν and τ for F_4 and G_2 as in Table 2. Also the orders of corresponding orbits are given in Table 2.

Type	E_6	E_7	E_8	F_4	G_2
ν	ω_2	ω_1	ω_8	ω_1	ω_2
τ	-	-	-	ω_4	ω_1
$\lvert W\nu\rvert$	72	126	240	24	6
$\lvert W\tau\rvert$	-	-	-	24	6

TABLE 2. Representatives of W-orbits

Let $V(\lambda)$ be the nontrivial irreducible representation with highest weight $\lambda \in \Lambda^+$. Let $m_\lambda(\mu)$ (abbr. $m(\mu)$) be the multiplicity of μ in $V(\lambda)$. We denote by $\Pi(\lambda)$ the set of weights occurring in $V(\lambda)$. Applying Freudenthal's formula [3, p.122] to $m_\lambda(0)$, we obtain

$$(M): \quad (\lambda, \lambda + 2\rho)m_\lambda(0) = \sum_{\alpha \in R}\sum_{k=1}^{\infty} m_\lambda(k\alpha)(\alpha, \alpha)k,$$

where $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha \ (= \sum_{i=1}^{n} \omega_i)$.

Lemma 2.1. *Let $\lambda = \sum_i m_i\omega_i$, $m_i \in \mathbb{Z}_+$, for each exceptional type. With the simple roots α_i described in Appendix, λ is described as follows.*

(1) *In case of E_6,*

$$\begin{aligned}\lambda &= (\tfrac{4}{3}m_1 + m_2 + \tfrac{5}{3}m_3 + 2m_4 + \tfrac{4}{3}m_5 + \tfrac{2}{3}m_6)\alpha_1\\ &+ (m_1 + 2m_2 + 2m_3 + 3m_4 + 2m_5 + m_6)\alpha_2\\ &+ (\tfrac{5}{3}m_1 + 2m_2 + \tfrac{10}{3}m_3 + 4m_4 + \tfrac{8}{3}m_5 + \tfrac{4}{3}m_6)\alpha_3\\ &+ (2m_1 + 3m_2 + 4m_3 + 6m_4 + 4m_5 + 2m_6)\alpha_4\\ &+ (\tfrac{4}{3}m_1 + 2m_2 + \tfrac{8}{3}m_3 + 4m_4 + \tfrac{10}{3}m_5 + \tfrac{5}{3}m_6)\alpha_5\\ &+ (\tfrac{2}{3}m_1 + m_2 + \tfrac{4}{3}m_3 + 2m_4 + \tfrac{5}{3}m_5 + \tfrac{4}{3}m_6)\alpha_6.\end{aligned}$$

(2) *In case of* E_7,

$$\begin{aligned}\lambda &= (2m_1 + 2m_2 + 3m_3 + 4m_4 + 3m_5 + 2m_6 + m_7)\alpha_1 \\ &+ (2m_1 + \tfrac{7}{2}m_2 + 4m_3 + 6m_4 + \tfrac{9}{2}m_5 + 3m_6 + \tfrac{3}{2}m_7)\alpha_2 \\ &+ (3m_1 + 4m_2 + 6m_3 + 8m_4 + 6m_5 + 4m_6 + 2m_7)\alpha_3 \\ &+ (4m_1 + 6m_2 + 8m_3 + 12m_4 + 9m_5 + 6m_6 + 3m_7)\alpha_4 \\ &+ (3m_1 + \tfrac{9}{2}m_2 + 6m_3 + 9m_4 + \tfrac{15}{2}m_5 + 5m_6 + \tfrac{5}{2}m_7)\alpha_5 \\ &+ (2m_1 + 3m_2 + 4m_3 + 6m_4 + 5m_5 + 4m_6 + 2m_7)\alpha_6 \\ &+ (m_1 + \tfrac{3}{2}m_2 + 2m_3 + 3m_4 + \tfrac{5}{2}m_5 + 2m_6 + \tfrac{3}{2}m_7)\alpha_7.\end{aligned}$$

(3) *In case of* E_8,

$$\begin{aligned}\lambda &= (4m_1 + 5m_2 + 7m_3 + 10m_4 + 8m_5 + 6m_6 + 4m_7 + 2m_8)\alpha_1 \\ &+ (5m_1 + 8m_2 + 10m_3 + 15m_4 + 12m_5 + 9m_6 + 6m_7 + 3m_8)\alpha_2 \\ &+ (7m_1 + 10m_2 + 14m_3 + 20m_4 + 16m_5 + 12m_6 + 8m_7 + 4m_8)\alpha_3 \\ &+ (10m_1 + 15m_2 + 20m_3 + 30m_4 + 24m_5 + 18m_6 + 12m_7 + 6m_8)\alpha_4 \\ &+ (8m_1 + 12m_2 + 16m_3 + 24m_4 + 20m_5 + 15m_6 + 10m_7 + 5m_8)\alpha_5 \\ &+ (6m_1 + 9m_2 + 12m_3 + 18m_4 + 15m_5 + 12m_6 + 8m_7 + 4m_8)\alpha_6 \\ &+ (4m_1 + 6m_2 + 8m_3 + 12m_4 + 10m_5 + 8m_6 + 6m_7 + 3m_8)\alpha_7 \\ &+ (2m_1 + 3m_2 + 4m_3 + 6m_4 + 5m_5 + 4m_6 + 3m_7 + 2m_8)\alpha_8.\end{aligned}$$

(4) *In case of* F_4,

$$\begin{aligned}\lambda &= (2m_1 + 3m_2 + 2m_3 + m_4)\alpha_1 \\ &+ (3m_1 + 6m_2 + 4m_3 + 2m_4)\alpha_2 \\ &+ (4m_1 + 8m_2 + 6m_3 + 3m_4)\alpha_3 \\ &+ (2m_1 + 4m_2 + 3m_3 + 2m_4)\alpha_4.\end{aligned}$$

(5) *In case of* G_2,

$$\lambda = (2m_1 + 3m_2)\alpha_1 + (m_1 + 2m_2)\alpha_2.$$

Proof. Let Ca be the Cartan matrix, which is given in Appendix. By the definition of ω_i, we have

$$(\alpha_1, \ldots, \alpha_l) = (\omega_1, \ldots, \omega_l)\, {}^tCa,$$

and thus

$$(\omega_1, \ldots, \omega_l) = (\alpha_1, \ldots, \alpha_l)\, {}^tCa^{-1},$$

where $l = 6, 7, 8, 4, 2$ according to E_l ($l = 6, 7, 8$), F_4 and G_2. This relation shows each description. □

Remark. In these computations, we used "Mathematica" a symbolic mathematical computation program developed by Wolfram.

Let $\lambda = \sum_i r_i\alpha_i \in \Lambda^+$, $r_i \in \mathbb{Q}_+$, with simple roots α_i. It is known that $m_\lambda(0) > 0$ if and only if $r_i \in \mathbb{Z}_+$. Thus we obtain the following from Lemma 2.1.

Lemma 2.2. *Let $\lambda = \sum_i m_i\omega_i$, $m_i \in \mathbb{Z}_+$, for each exceptional type.*

(1) *In case of E_6, $m_\lambda(0) > 0$ if and only if $m_1 + m_5 \equiv m_3 + m_6 \bmod 3$.*
(2) *In case of E_7, $m_\lambda(0) > 0$ if and only if $m_2 + m_5 + m_7 \equiv 0 \bmod 2$.*
(3) *In cases of E_8, F_4 and G_2, $m_\lambda(0) > 0$ for any $\lambda \in \Lambda^+$.*

For $\mu \in \Pi(\lambda)$, we set

$$n_\mu = \max\{k \geq 1 \mid m(k\mu) \neq 0\}.$$

If $\mu \notin \Pi(\lambda)$, then we set $n_\mu = 0$. Set

$$K_\lambda = (\lambda, \lambda + 2\rho), \quad A_\lambda = \sum_{k=1}^{n_\nu} m(k\nu), \quad B_\lambda = \sum_{k=1}^{n_\tau} m(k\tau).$$

The following result is described in [4] without the detailed proof.

Lemma 2.3. *For each exceptional type, if $m(0) > 0$, then $n_\nu = [a_\lambda]$ and $n_\tau = [b_\lambda]$, where $[x]$ denotes the greatest integer not exceeding x, where a_λ and b_λ are listed below.*

(1) $E_6 : a_\lambda = \frac{1}{2}m_1 + m_2 + m_3 + \frac{3}{2}m_4 + m_5 + \frac{1}{2}m_6.$
(2) $E_7 : a_\lambda = m_1 + m_2 + \frac{3}{2}m_3 + 2m_4 + \frac{3}{2}m_5 + m_6 + \frac{1}{2}m_7.$
(3) $E_8 : a_\lambda = m_1 + \frac{3}{2}m_2 + 2m_3 + 3m_4 + \frac{5}{2}m_5 + 2m_6 + \frac{3}{2}m_7 + m_8.$
(4) $F_4 : a_\lambda = m_1 + \frac{3}{2}m_2 + m_3 + \frac{1}{2}m_4, \quad b_\lambda = m_1 + 2m_2 + \frac{3}{2}m_3 + m_4.$
(5) $G_2 : a_\lambda = \frac{1}{2}m_1 + m_2, \quad b_\lambda = m_1 + \frac{3}{2}m_2.$

Proof. These are obtained from the following descriptions.

(1) Since $\nu = \omega_2$, we have

$$\begin{aligned}
\lambda - k\omega_2 = &(-k + \tfrac{4}{3}m_1 + m_2 + \tfrac{5}{3}m_3 + 2m_4 + \tfrac{4}{3}m_5 + \tfrac{2}{3}m_6)\alpha_1 \\
&+ 2(-k + \tfrac{1}{2}m_1 + m_2 + m_3 + \tfrac{3}{2}m_4 + m_5 + \tfrac{1}{2}m_6)\alpha_2 \\
&+ 2(-k + \tfrac{5}{6}m_1 + m_2 + \tfrac{5}{3}m_3 + 2m_4 + \tfrac{4}{3}m_5 + \tfrac{2}{3}m_6)\alpha_3 \\
&+ 3(-k + \tfrac{2}{3}m_1 + m_2 + \tfrac{4}{3}m_3 + 2m_4 + \tfrac{4}{3}m_5 + \tfrac{2}{3}m_6)\alpha_4 \\
&+ 2(-k + \tfrac{2}{3}m_1 + m_2 + \tfrac{4}{3}m_3 + 2m_4 + \tfrac{5}{3}m_5 + \tfrac{5}{6}m_6)\alpha_5 \\
&+ (-k + \tfrac{2}{3}m_1 + m_2 + \tfrac{4}{3}m_3 + 2m_4 + \tfrac{5}{3}m_5 + \tfrac{4}{3}m_6)\alpha_6.
\end{aligned}$$

(2) Since $\nu = \omega_1$, we have

$$\begin{aligned}
\lambda - k\omega_1 = &2(-k + m_1 + m_2 + \tfrac{3}{2}m_3 + 2m_4 + \tfrac{3}{2}m_5 + m_6 + \tfrac{1}{2}m_7)\alpha_1 \\
&+ 2(-k + m_1 + \tfrac{7}{4}m_2 + 2m_3 + 3m_4 + \tfrac{9}{4}m_5 + \tfrac{3}{2}m_6 + \tfrac{3}{4}m_7)\alpha_2 \\
&+ 3(-k + m_1 + \tfrac{4}{3}m_2 + 2m_3 + \tfrac{8}{3}m_4 + 2m_5 + \tfrac{4}{3}m_6 + \tfrac{2}{3}m_7)\alpha_3 \\
&+ 4(-k + m_1 + \tfrac{3}{2}m_2 + 2m_3 + 3m_4 + \tfrac{9}{4}m_5 + 3m_6 + \tfrac{3}{4}m_7)\alpha_4 \\
&+ 3(-k + m_1 + \tfrac{3}{2}m_2 + 2m_3 + 3m_4 + \tfrac{5}{2}m_5 + \tfrac{5}{3}m_6 + \tfrac{5}{6}m_7)\alpha_5 \\
&+ 2(-k + m_1 + \tfrac{3}{2}m_2 + 2m_3 + 3m_4 + \tfrac{5}{2}m_5 + 2m_6 + m_7)\alpha_6 \\
&+ (-k + m_1 + \tfrac{3}{2}m_2 + 2m_3 + 3m_4 + \tfrac{5}{2}m_5 + 2m_6 + \tfrac{3}{2}m_7)\alpha_7.
\end{aligned}$$

(3) Since $\nu = \omega_8$, we have

$$\begin{aligned}
\lambda - k\omega_8 = &2(-k + 2m_1 + \tfrac{5}{2}m_2 + \tfrac{7}{2}m_3 + 5m_4 + 4m_5 + 3m_6 + 2m_7 + m_8)\alpha_1 \\
&+ 3(-k + \tfrac{5}{3}m_1 + \tfrac{8}{3}m_2 + \tfrac{10}{3}m_3 + 5m_4 + 4m_5 + 3m_6 + 2m_7 + m_8)\alpha_2 \\
&+ 4(-k + \tfrac{7}{4}m_1 + \tfrac{5}{2}m_2 + \tfrac{7}{2}m_3 + 5m_4 + 4m_5 + 3m_6 + 2m_7 + m_8)\alpha_3 \\
&+ 6(-k + \tfrac{5}{3}m_1 + \tfrac{5}{2}m_2 + \tfrac{10}{3}m_3 + 5m_4 + 4m_5 + 3m_6 + 2m_7 + m_8)\alpha_4 \\
&+ 5(-k + \tfrac{8}{5}m_1 + \tfrac{12}{5}m_2 + \tfrac{16}{5}m_3 + \tfrac{24}{5}m_4 + 4m_5 + 3m_6 + 2m_7 + m_8)\alpha_5 \\
&+ 4(-k + \tfrac{3}{2}m_1 + \tfrac{9}{4}m_2 + 3m_3 + \tfrac{9}{2}m_4 + \tfrac{15}{4}m_5 + 3m_6 + 2m_7 + m_8)\alpha_6 \\
&+ 3(-k + \tfrac{4}{3}m_1 + 2m_2 + \tfrac{8}{3}m_3 + 4m_4 + \tfrac{10}{3}m_5 + \tfrac{8}{3}m_6 + 2m_7 + m_8)\alpha_7 \\
&+ 2(-k + m_1 + \tfrac{3}{2}m_2 + 2m_3 + 3m_4 + \tfrac{5}{2}m_5 + 2m_6 + \tfrac{3}{2}m_7 + m_8)\alpha_8.
\end{aligned}$$

(4) Since $\nu = \omega_1$, we have

$$\begin{aligned}\lambda - k\omega_1 &= 2(-k + m_1 + \tfrac{3}{2}m_2 + m_3 + \tfrac{1}{2}m_4)\alpha_1 \\ &\quad + 3(-k + m_1 + 2m_2 + \tfrac{4}{3}m_3 + \tfrac{2}{3}m_4,)\alpha_2 \\ &\quad + 4(-k + m_1 + 2m_2 + \tfrac{3}{2}m_3 + \tfrac{3}{4}m_4)\alpha_3 \\ &\quad + 2(-k + m_1 + 2m_2 + \tfrac{3}{2}m_3 + m_4)\alpha_4.\end{aligned}$$

and since $\tau = \omega_4$, we have

$$\begin{aligned}\lambda - k\omega_4 &= (-k + 2m_1 + 3m_2 + 2m_3 + m_4)\alpha_1 \\ &\quad + 2(-k + \tfrac{3}{2}m_1 + 3m_2 + 2m_3 + m_4)\alpha_2 \\ &\quad + 3(-k + \tfrac{4}{3}m_1 + \tfrac{8}{3}m_2 + 2m_3 + m_4)\alpha_3 \\ &\quad + 2(-k + m_1 + 2m_2 + \tfrac{3}{2}m_3 + m_4)\alpha_4.\end{aligned}$$

(5) Since $\nu = \omega_2$ and $\tau = \omega_1$, we have

$$\begin{aligned}\lambda - k\omega_2 &= 3(-k + \tfrac{2}{3}m_1 + m_2)\alpha_1 + 2(-k + \tfrac{1}{2}m_1 + m_2)\alpha_2, \\ \lambda - k\omega_1 &= 2(-k + m_1 + \tfrac{3}{2}m_2)\alpha_1 + (-k + m_1 + 2m_2)\alpha_2.\end{aligned}$$

□

We recall the following result proved in [4].

Lemma 2.4. *With the notation above,*

(1) *For each simply laced type, the following inequality holds*

$$r(V(\lambda)) \leq \frac{2a_\lambda}{K_\lambda + 2a_\lambda} \quad (\lambda \in \Lambda^+ \smallsetminus \{0\}).$$

(2) *For each nonsimply laced type, the following inequality holds.*

$$r(V(\lambda)) \leq \frac{(\nu,\nu)a_\lambda \frac{|W\nu|}{|R|} + (\tau,\tau)b_\lambda \frac{|W\tau|}{|R|}}{K_\lambda + (\nu,\nu)a_\lambda \frac{|W\nu|}{|R|} + (\tau,\tau)b_\lambda \frac{|W\tau|}{|R|}} \quad (\lambda \in \Lambda^+ \smallsetminus \{0\}).$$

3. **Proof in the case of E_l**

We set $\lambda = \sum_{i=1}^{l} m_i\omega_i$ as before.

The case of E_6: By Lemma 2.4, it suffices to show that $\dfrac{2a_\lambda}{K_\lambda + 2a_\lambda} \leq \dfrac{1}{13}$, which is equivalent to $K_\lambda - 24a_\lambda \geq 0$. Indeed, we see

$$\begin{aligned} K_\lambda - 24a_\lambda &= 2m_2^2 - 2m_2 + 4m_1 + \tfrac{4}{3}m_1^2 + 2m_1m_2 + 6m_3 + \tfrac{10}{3}m_1m \\ &\quad + 4m_2m_3 + \tfrac{10}{3}m_3^2 + 6m_4 + 4m_1m_4 + 6m_2m_4 + 8m_3m_4 \\ &\quad + 6m_4^2 + 6m_5 + \tfrac{8}{3}m_1m_5 + 4m_2m_5 + \tfrac{16}{3}m_3m_5 \\ &\quad + 8m_4m_5 + \tfrac{10}{3}m_5^2 + 4m_6 + \tfrac{4}{3}m_1m_6 \\ &\quad + 2m_2m_6 + \tfrac{8}{3}m_3m_6 + 4m_4m_6 + \tfrac{10}{3}m_5m_6 + \tfrac{4}{3}m_6^2 \\ &= 2m_2(m_2 - 1) + \text{nonnegative terms} \geq 0. \end{aligned}$$

If $\lambda = \omega_2$, then V_λ is the adjoint representation and $r(V_\lambda) = 1/13$. Thus $d_G = 1/13$.

Similarly, for E_7, we may show $K_\lambda - 36a_\lambda \geq 0$. Indeed,

$$\begin{aligned} K_\lambda - 36a_\lambda &= 2m_1^2 - 2m_1 + 13m_2 + 4m_1m_2 + \tfrac{7}{2}m_2^2 + 12m_3 + 6m_1m_3 \\ &\quad + 8m_2m_3 + 6m_3^2 + 24m_4 + 8m_1m_4 + 12m_2m_4 + 16m_3m_4 \\ &\quad + 12m_4^2 + 21m_5 + 6m_1m_5 + 9m_2m_5 + 12m_3m_5 + 18m_4m_5 \\ &\quad + \tfrac{15}{2}m_5^2 + 16m_6 + 4m_1m_6 + 6m_2m_6 + 8m_3m_6 + 12m_4m_6 \\ &\quad + 10m_5m_6 + 4m_6^2 + 9m_7 + 2m_1m_7 + 3m_2m_7 + 4m_3m_7 \\ &\quad + 6m_4m_7 + 5m_5m_7 + 4m_6m_7 + \tfrac{3}{2}m_7^2 \\ &= 2m_1(m_1 - 1) + \text{nonnegative terms} \geq 0. \end{aligned}$$

If $\lambda = \omega_1$, then we see $r(V_\lambda) = 1/19$; thus $d_G = 1/19$.

For E_8, we may show $K_\lambda - 60a_\lambda \geq 0$. Indeed,

$$\begin{aligned} K_\lambda - 60a_\lambda &= 2m_8^2 - 2m_8 + 32m_1 + 4m_1^2 + 46m_2 + 10m_1m_2 + 8m_2^2 + 62m_3 \\ &\quad + 14m_1m_3 + 20m_2m_3 + 14m_3^2 + 90m_4 + 20m_1m_4 + 30m_2m_4 \\ &\quad + 40m_3m_4 + 30m_4^2 + 70m_5 + 16m_1m_5 + 24m_2m_5 + 32m_3m_5 \\ &\quad + 48m_4m_5 + 20m_5^2 + 48m_6 + 12m_1m_6 + 18m_2m_6 + 24m_3m_6 \\ &\quad + 36m_4m_6 + 30m_5m_6 + 12m_6^2 + 24m_7 + 8m_1m_7 + 12m_2m_7 \\ &\quad + 16m_3m_7 + 24m_4m_7 + 20m_5m_7 + 16m_6m_7 + 6m_7^2 \\ &\quad + 4m_1m_8 + 6m_2m_8 + 8m_3m_8 + 12m_4m_8 + 10m_5m_8 \\ &\quad + 8m_6m_8 + 6m_7m_8 \\ &= 2m_8(m_8 - 1) + \text{nonnegative terms} \geq 0. \end{aligned}$$

If $\lambda = \omega_8$, then we see $r(V_\lambda) = 1/31$; thus $d_G = 1/31$.

4. **Proof in the cases of F_4 and G_2**

The case of F_4: By Lemma 2.4, it suffices to show that

$$\frac{a_\lambda + \frac{1}{2}b_\lambda}{K_\lambda + a_\lambda + \frac{1}{2}b_\lambda} \leq \frac{1}{13},$$

which is equivalent to $K_\lambda - 12a_\lambda - 6b_\lambda \geq 0$. Indeed

$$\begin{aligned} K_\lambda - 12a_\lambda - 6b_\lambda &= -2m_1 + 2m_1^2 + 6m_1m_2 + 6m_2^2 + 4m_1m_3 + 8m_2m_3 \\ &\quad + 3m_3^2 + 2m_1m_4 + 4m_2m_4 + 3m_3m_4 - m_4 + m_4^2 \\ &= 2m_1(m_1 - 1) + m_4(m_4 - 1) + \text{nonnegative terms} \geq 0. \end{aligned}$$

The case of G_2: In this case, we have

$$\begin{aligned} K_\lambda - 18a_\lambda - 6b_\lambda &= -5m_1 + 2m_1^2 - 9m_2 + 6m_1m_2 + 6m_2^2 \\ &= 2m_1(m_1 - 5/2) + 6m_2(m_1 + m_2 - 3/2). \end{aligned}$$

If $(m_1, m_2) \neq (2, 0), (1, 1), (1, 0)$, then $K_\lambda - 18a_\lambda - 6b_\lambda \geq 0$. This means $r(V_\lambda) \leq 1/7$ by Lemma 2.4. If $(m_1, m_2) = (2, 0), (1, 1), (1, 0)$, then Freudenthal's multiplicity formula shows that $r(V_\lambda) = 1/9, 1/7, 1/7$ respectively. Thus we have $d_G = 1/7$. Note that $\lambda = \omega_1$ represents the adjoint representation.

Appendix — Simple roots and dominant weights

For convenience to the readers, we summarize well-known facts: simple roots, dominant weights and related necessary data for each exceptional type, see [1, 2, 3] for the details. Let e_i denote the i-th fundamental unit vector in a euclidean space $\mathbb{R}^m$ with standard inner product, where $m = 8$ for E_l ($l = 6, 7, 8$), $m = 4$ for F_4 and $m = 3$ for G_2.

(1) E_6.

Simple roots: $\alpha_1 = \frac{1}{2}(e_1 - \sum_{i=2}^{7} e_i + e_8)$, $\alpha_2 = e_1 + e_2$, $\alpha_i = -e_{i-2} + e_{i-1}$ $(3 \leq i \leq 6)$.

Positive roots: $\alpha_\varepsilon = \frac{1}{2}(\sum_{i=1}^{5}(-1)^{\varepsilon_i}e_i - e_6 - e_7 + e_8)$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_5) \in \{0, 1\}^5$ with $\sum_{i=1}^{5} \varepsilon_i$ even, $\alpha'_{ij} = -e_i + e_j$, $\beta_{ij} = e_i + e_j$ $(1 \leq i < j \leq 5)$

Dominant weights: $\lambda = \sum_{i=1}^{6} m_i\omega_i$ with $m_i \in \mathbb{Z}_+$ and the fundamental weights $\omega_1 = \frac{2}{3}(-e_6 - e_7 + e_8)$, $\omega_2 = \alpha_{(0,\ldots,0)} = \frac{1}{2}(\sum_{i=1}^{5} e_i - e_6 - e_7 + e_8)$,

$\omega_3 = \frac{1}{2}(-e_1 + \sum_{i=2}^{5} e_i) + \frac{5}{6}(-e_6 - e_7 + e_8)$, $\omega_k = \sum_{i=k-1}^{5} e_i + \frac{7-k}{3}(-e_6 - e_7 + e_8)$, $k = 4,\ 5,\ 6$.

The half sum of positive roots: $\rho = \sum_{n=1}^{6} \omega_i$.

Cartan matrix and its inverse:

$$Ca = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2, & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$Ca^{-1} = \begin{pmatrix} 4/3 & 1 & 5/3 & 2 & 4/3 & 2/3 \\ 1 & 2 & 2 & 3 & 2 & 1 \\ 5/3 & 2 & 10/3 & 4 & 8/3 & 4/3 \\ 2 & 3 & 4 & 6 & 4 & 2 \\ 4/3 & 2 & 8/3 & 4 & 10/3 & 5/3 \\ 2/3 & 1 & 4/3 & 2 & 5/3 & 4/3 \end{pmatrix}.$$

(2) E_7.

Simple roots: $\alpha_1 = \frac{1}{2}(e_1 - \sum_{i=2}^{7} e_i + e_8)$, $\alpha_2 = e_1 + e_2$, $\alpha_i = -e_{i-2} + e_{i-1}$ $(3 \leq i \leq 7)$.

Positive roots: $\alpha_\varepsilon = \frac{1}{2}(\sum_{i=1}^{6}(-1)^{\varepsilon_i} e_i - e_7 + e_8)$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_6) \in \{0,1\}^6$ with $\sum_{i=1}^{6} \varepsilon_i$ odd, $\alpha'_{ij} = -e_i + e_j$, $\beta_{ij} = e_i + e_j$ $(1 \leq i < j \leq 6)$, $\alpha'_{78} = -e_7 + e_8$.

Dominant weights: $\lambda = \sum_{i=1}^{7} m_i \omega_i$ with $m_i \in \mathbb{Z}_+$ and the fundamental weights $\omega_1 = \alpha'_{78} = -e_7 + e_8$, $\omega_2 = \frac{1}{2}\sum_{i=1}^{6} e_i - e_7 + e_8$, $\omega_3 = \frac{1}{2}(-e_1 + \sum_{i=2}^{6} e_i) + \frac{3}{2}(-e_7 + e_8)$, $\omega_k = \sum_{i=k-1}^{6} e_i + \frac{8-k}{2}(-e_7 + e_8)$, $k = 4,\ 5,\ 6,\ 7$.

The half sum of positive roots: $\rho = \sum_{n=1}^{7} \omega_i$.

Cartan matrix and its inverse:

$$Ca = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$Ca^{-1} = \begin{pmatrix} 2 & 2 & 3 & 4 & 3 & 2 & 1 \\ 2 & 7/2 & 4 & 6 & 9/2 & 3 & 3/2 \\ 3 & 4 & 6 & 8 & 6 & 4 & 2 \\ 4 & 6 & 8 & 12 & 9 & 6 & 3 \\ 3 & 9/2 & 6 & 9 & 15/2 & 5 & 5/2 \\ 2 & 3 & 4 & 6 & 5 & 4 & 2 \\ 1 & 3/2 & 2 & 3 & 5/2 & 2 & 3/2 \end{pmatrix}.$$

(3) E_8.

Simple roots: $\alpha_1 = \frac{1}{2}(e_1 - \sum_{i=2}^{7} e_i + e_8)$, $\alpha_2 = e_1 + e_2$, $\alpha_i = -e_{i-2} + e_{i-1}$ $(3 \leq i \leq 8)$.

Positive roots: $\alpha_\varepsilon = \frac{1}{2}(\sum_{i=1}^{7}(-1)^{\varepsilon_i} e_i + e_8)$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_7) \in \{0,1\}^7$ with $\sum_{i=1}^{7} \varepsilon_i$ even, $\alpha'_{ij} = -e_i + e_j$, $\beta_{ij} = e_i + e_j$ $(1 \leq i < j \leq 8)$.

Dominant weights: $\lambda = \sum_{i=1}^{8} m_i \omega_i$ with $m_i \in \mathbb{Z}_+$ and the fundamental weights $\omega_1 = 2e_8$, $\omega_2 = \frac{1}{2}\sum_{i=1}^{7} e_i + \frac{5}{2}e_8$, $\omega_3 = \frac{1}{2}(-e_1 + \sum_{i=2}^{7} e_i) + \frac{7}{2}e_8$, $\omega_k = \sum_{i=k-1}^{7} e_i + (9-k)e_8$, $k = 4, 5, 6, 7, 8$.

The half sum of positive roots: $\rho = \sum_{n=1}^{8} \omega_i$.

Cartan matrix and its inverse:

$$Ca = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$Ca^{-1} = \begin{pmatrix} 4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\ 5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\ 7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\ 10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\ 8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\ 6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\ 4 & 6 & 8 & 12 & 10 & 8 & 6 & 3 \\ 2 & 3 & 4 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}.$$

(4) F_4.

Simple roots: $\alpha_1 = e_2 - e_3$, $\alpha_2 = e_3 - e_4$, $\alpha_3 = e_4$, $\alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$.

Positive roots: $\alpha_{ij} = e_i - e_j$, $\beta_{ij} = e_i + e_j$ $(1 \le i < j \le 4)$. $\gamma_i = e_i$ $(1 \le i \le 4)$, $\alpha_\varepsilon = \frac{1}{2}(e_1 + (-1)^{\varepsilon_1} e_2 + (-1)^{\varepsilon_2} e_3 + (-1)^{\varepsilon_3} e_4)$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{0, 1\}^3$.

Dominant weights: $\lambda = \sum_{i=1}^4 m_i \omega_i$ with $m_i \in \mathbb{Z}_+$ and the fundamental weights $\omega_1 = e_1 + e_2$, $\omega_2 = 2e_1 + e_2 + e_3$, $\omega_3 = \frac{1}{2}(3e_1 + e_2 + e_3 + e_4)$, $\omega_4 = e_1$.

The half sum of positive roots: $\rho = \sum_{n=1}^4 \omega_i$.

Cartan matrix and its inverse:

$$Ca = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \qquad Ca^{-1} = \begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 6 & 8 & 4 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 2 \end{pmatrix}.$$

(5) G_2.

Simple roots: $\alpha_1 = e_1 - e_2$, $\alpha_2 = -2e_1 + e_2 + e_3$.

Positive roots: $\beta_1 = e_1 - e_2$ $(= \alpha_1)$, $\beta_2 = -e_1 + e_3$, $\beta_3 = -e_2 + e_3$, $\delta_1 = -2e_1 + e_2 + e_3$ $(= \alpha_2)$, $\delta_2 = e_1 - 2e_2 + e_3$, $\delta_3 = -e_1 - e_2 + 2e_3$

Dominant weights: $\lambda = m_1\omega_1 + m_2\omega_2$ with $m_i \in \mathbb{Z}_+$ and the fundamental weights $\omega_1 = \beta_3 = -e_2 + e_3$, $\omega_2 = \delta_3 = -e_1 - e_2 + 2e_3$

The half sum of positive roots: $\rho = \omega_1 + \omega_2$.

Cartan matrix and its inverse:

$$Ca = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \qquad Ca^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

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