Computation of the K-theoretic Euler classes of representations for a non-abelian group of order p^3 and exponent p

Ikumitsu NAGASAKI [1)]

Abstract. For an odd prime p, there exists a unique non-abelian group of order p^3 and exponent p, denoted by M_p. In this article, we shall describe the structure of the representation ring and compute the K-theoretic Euler classes of M_p-representations.

1. Background

Let G be a compact Lie group. The K-theoretic Euler classes of G-representations are utilized to obtain Borsuk-Ulam type results for G-maps, see [2], [7], [9], etc. A key result is the following proposition due to Atiyah-Tall [1].

Proposition 1.1 ([1]). *Let V and W be unitary G-representations. If there exits a G-map $h : S(V) \to S(W)$, then*

$$e_G(W) = z(f)e_G(V)$$

for some $z(f) \in R(G)$, where $e_G(\ )$ denotes the K-theoretic Euler class of a representation.

However, the preceding researches treat only abelian compact Lie groups, e.g, finite cyclic groups, circle groups and their products. In this article, we treat a non-abelian group M_p of order p^3 and exponent p which is given by

$$M_p = \{a, b, c \,|\, ba = abc,\ ac = ca,\ bc = cb\},\ \text{see [6]}.$$

This group plays an important role in determining finite groups with the Borsuk-Ulam property, see [8]. For other applications, it seems to be worth computing the K-theoretic Euler classes of M_p-representations.

1) Department of Mathematics, Kyoto Prefectural University of Medicine, 1-5 Shimogamohangi-cho, Sakyo-ku, Kyoto 606-0823, Japan. e-mail: nagasaki@koto.kpu-m.ac.jp

2010 *Mathematics Subject Classification.* Primary 55N15; Secondary 57S17.

2. The group structure of M_p

In this section, we describe subgroups, their normalizers and conjugacy classes of $G = M_p$. This group is known as a unique non-abelian finite group of order p^3 and exponent p, see [6]. Note that $c = b^{-1}a^{-1}ba$ and $C = \langle c\rangle$ is the center of M_p. Any element g of M_p is uniquely described as $g = a^s b^t c^u$ for some $s, t, u \in \mathbb{F}_p$, where $\mathbb{F}_p$ is the finite field of order p.

2.0.1. *Conjugacy classes of elements.* There are p^2+p-1 conjugacy classes of elements:

$$(a^s b^t) = \{a^s b^t, a^s b^t c, \dots, a^s b^t c^{p-1}\}$$

for $(s, t) \neq (0, 0)$, and

$$(1), (c), \dots, (c^{p-1}).$$

2.0.2. *Subgroups.* The subgroups of M_p are listed as follows.

(1) Order 1. $\{1\}$.

(2) There are $p^2 + p + 1$ subgroups of order p:

$$\begin{array}{ccccc} \langle c\rangle & & & & \\ \langle a\rangle & \langle ab\rangle & \cdots & \langle ab^{p-1}\rangle & \langle b\rangle \\ \vdots & & & & \vdots \\ \langle ac^{p-1}\rangle & \langle abc^{p^1}\rangle & \cdots & \langle ab^{p-1}c^{p-1}\rangle & \langle bc^{p-1}\rangle \end{array}$$

(3) There are $p + 1$ subgroups of order p^2:

$$\langle a, c\rangle, \quad \langle ab, c\rangle, \quad \cdots, \quad \langle ab^{p-1}, c\rangle, \quad \langle b, c\rangle.$$

(4) Order p^3. $G = M_p$.

2.0.3. *Normalizers of subgroups.* The normal subgroups are 1, $\langle c\rangle$, the subgroups of order p^2 and G.

In other subgroups, $N_G(\langle ab^t c^u\rangle) = \langle ab^t, c\rangle$ for $t, u \in \mathbb{F}_p$, and $N_G(\langle bc^u\rangle) = \langle b, c\rangle$ for $u \in \mathbb{F}_p$.

2.0.4. *Conjugacy classes of subgroups.* There are $2p+5$ conjugacy classes of subgroups. The representatives are listed as follows.

$$\{1\}, \quad \langle a\rangle, \quad \langle ab\rangle, \quad \cdots \quad , \langle ab^{p-1}\rangle, \quad \langle b\rangle, \quad \langle c\rangle,$$

$$\langle a, c\rangle, \quad \langle ab, c\rangle, \quad \cdots \quad , \langle ab^{p-1}, c\rangle, \quad \langle b, c\rangle, \quad G.$$

We set

$$K_k = \langle ab^k \rangle, \quad H_k = \langle ab^k, c \rangle \ \ (0 \leq k \leq p-1), \ K_p = \langle b \rangle, \quad H_p = \langle b, c \rangle.$$

3. The representation ring of M_p

Let G be M_p as before. In this section, we determine the complex representation ring $R(G)$ of G. First, we give the irreducible G-representations. Since G has a split extension

$$1 \to \langle a, c \rangle \to G \to \langle \bar{b} \rangle \to 1,$$

where $\bar{b} = q(b)$, $q : G \to G/\langle a, c \rangle$ the projection, we can obtain the irreducible representations by the method described in [10]. We set $H = H_0 = \langle a, c \rangle$ and $K = K_0 = \langle \bar{b} \rangle$. Let $X = \mathrm{Hom}(H, \mathbb{C}^*)$, the ring of 1-dimensional irreducible H-representations (or equivalently the character ring of H). Since $H \cong C_p \times C_p$, X consists of the irreducible representations $S_{k,l}$ whose characters $\psi_{k,l}$ are defined by

$$\psi_{k,l}(a) = \xi_p^k, \quad \psi_{k,l}(c) = \xi_p^l,$$

for $k, l \in \mathbb{F}_p$, where ξ_p is a primitive p-th root of unity. Then K acts on X by

$${}^{\bar{b}}\psi_{k,l}(h) = \psi_{k,l}(b^{-1}hb), \quad h \in H.$$

Letting $h = a^s c^u$, we see $b^{-1}hb = a^s c^{u-s}$ and $\psi_{k,l}(b^{-1}hb) = \xi_p^{s(k-l)+ul}$; hence we obtain

$${}^{\bar{b}}\psi_{k,l} = \psi_{k-l,l}.$$

Therefore the representatives of the orbit set X/K are taken to be

$$\psi_{k,0}, \ k \in \mathbb{F}_p \text{ (fixed points), and } \psi_{0,l}, \ l \in \mathbb{F}_p \smallsetminus \{0\}.$$

Then $\psi_{k,0}$ can be extended to a function on G by $\psi_{k,0}(a^s b^t c^u) = \psi_{k,0}(a^s c^u) = \xi_p^{sk}$, which gives a 1-dimensional G-representation, denoted by S_k. On the other hand, there are 1-dimensional irreducible G-representations T_l, $l \in \mathbb{F}_p$, by lifting the 1-dimensional irreducible K-representations whose characters are given by $\tau_l(\bar{b}) = \xi_p^l$. Thus we have p^2 1-dimensional G-representation $V_{k,l} := S_k \otimes T_l$. The character $\chi_{k,l}$ of $V_{k,l}$ is given by

$$\chi_{k,l}(a) = \xi_p^k, \ \chi_{k,l}(b) = \xi_p^l \ \text{ and } \ \chi_{k,l}(c) = 1.$$

Eventually, $V_{k,l}$ is isomorphic to the lifting of the irreducible representation $\overline{V}_{k,l}$ of $G/\langle c \rangle = \langle \bar{a}, \bar{b} \rangle \cong C_p \times C_p$ whose character $\overline{\chi}_{k,l}$ is given by $\overline{\chi}_{k,l}(\bar{a}) = \xi_p^k$ and $\overline{\chi}_{k,l}(\bar{b}) = \xi_p^l$

Next consider the H-representation $S_{0,m}$ whose character is $\psi_{0,m}$, $m \in \mathbb{F}_p \setminus \{0\}$. Then we have a p-dimensional irreducible representation $U_m := \mathrm{Ind}_H^G S_{0,m}$. The character χ_m of U_m is given by

$$\chi_m(a^s b^t c^u) = \begin{cases} 0 & (s \neq 0 \text{ or } t \neq 0) \\ p\xi_p^{mu} & (s = t = 0). \end{cases}$$

Consequently we have

Proposition 3.1. *Let $G = M_p$. The irreducible G-representations are given by*

(1) *$V_{k,l}$ for $k, l \in \mathbb{F}_p$, whose character $\chi_{k,l}$ is given by*

$$\chi_{k,l}(a^s b^t c^u) = \xi_p^{sk+tl},$$

and

(2) *U_m for $m \in \mathbb{F}_p^* := \mathbb{F}_p \setminus \{0\}$, whose character χ_m is given by*

$$\chi_m(a^s b^t c^u) = \begin{cases} 0 & (s \neq 0 \text{ or } t \neq 0) \\ p\xi_p^{mu} & (s = t = 0). \end{cases}$$

Hence $R(G)$ is isomorphic to $\mathbb{Z}^{p^2+p-1}$ as additive groups.

Next we determine the ring structure of $R(G)$.

Proposition 3.2. *With notation as above,*

(1) $V_{1,0}^{\otimes p} \cong \mathbb{C}$, $V_{0,1}^{\otimes p} \cong \mathbb{C}$,
(2) $V_{k,l} \cong V_{1,0}^{\otimes k} \otimes V_{0,1}^{\otimes l}$,
(3) $V_{k,l} \otimes V_{k',l'} \cong V_{k+k',l+l'}$,
(4) $V_{k,l} \otimes U_m \cong U_m$, *and*
(5) $U_m \otimes U_n \cong \begin{cases} pU_{m+n} & m+n \neq 0 \text{ in } \mathbb{F}_p \\ \oplus_{i,j \in \mathbb{F}_p} V_{i,j} & m+n = 0 \text{ in } \mathbb{F}_p \end{cases}$

Proof. It suffices to show that the characters of both sides coincide. Let $g = a^s b^t c^u$.

(1) is clear.

(2) and (3) follows from

$$\chi_{k,l}(g)\chi_{k',l'}(g) = \xi_p^{sk+tl}\xi_p^{sk'+tl'} = \xi_p^{s(k+k')+t(l+l')} = \chi_{k+k',l+l'}(g).$$

(4) If $s \neq 0$ or $t \neq 0$, then $\chi_m(g) = 0$; hence $\chi_{k,l}(g)\chi_m(g) = \chi_m(g)$. If $s = t = 0$, then $\chi_{k,l}(c^u) = 1$; hence $\chi_{k,l}(c^u)\chi_m(c^u) = \chi_m(c^u)$. Thus $\chi_{k,l}\chi_m = \chi_m$.

(5) In the case where $m+n \neq 0$ in $\mathbb{F}_p$, if $s \neq 0$ or $t \neq 0$, then $\chi_m(g)\chi_n(g) = 0 = p\chi_{m+n}(g)$. If $s = t = 0$, then $\chi_m(c^u)\chi_n(c^u) = p^2\xi_p^{(m+n)u} = p\chi_{m+n}(c^u)$; thus $\chi_m\chi_n = p\chi_{m+n}$.

In the case where $m+n = 0$ in $\mathbb{F}_p$, set $\psi = \sum_{i,j}\chi_{i,j}$ the character of $\oplus_{i,j}V_{i,j}$. When $s = t = 0$, we see $\chi_m(c^u)\chi_n(c^u) = p^2 = \psi(c^u)$. If $s \neq 0$ or $t \neq 0$, then $\psi(g) = (\sum_i \xi_p^{ki})(\sum_j \xi_p^{lj}) = 0$. Thus $\chi_m\chi_n = \psi$. □

Set $x = V_{1,0}$, $y = V_{0,1}$ and $z_m = U_m$, $m \in \mathbb{F}_p^*$ in $R(G)$. Let R be a polynomial ring

$$R = \mathbb{Z}[x, y, z_1, z_2, \dots, z_{p-1}].$$

Consider the ideal I generated by the following elements in R:

(1) $x^p - 1$, $y^p - 1$,

(2) $xz_m - z_m$, $yz_m - z_m$ for $m \in \mathbb{F}_p^*$,

(3) $z_m z_n - pz_{n+m}$ for $m+n \neq 0$, $m, n \in \mathbb{F}_p^*$,

(4) $z_m z_n - \sum_{i,j\in\mathbb{F}_p} x^i y^j$ for $m+n = 0$, $m, n \in \mathbb{F}_p^*$.

By Proposition 3.2, we obtain

Proposition 3.3. *As commutative rings, $R(G)$ is isomorphic to R/I.*

4. The K-theoretic Euler classes of M_p-representations

In equivariant K-theory, there exists the Thom isomorphism

$$\Phi : K_G(X) := K_G(X^+, +) \to K_G(E) := K_G(D(E), S(E))$$

for any complex G-vector bundle over a locally compact based G-space, where X^+ is the one-point compactification of X with infinite point $+$.

We consider the case of $X = *$ a one-point space. By definition, the equivariant complex K-group $K_G(*)$ is naturally isomorphic to the representation ring $R(G)$ as commutative rings. Let $\tau_V = \Phi(1) \in K_G(V)$, called the Thom class, where $1 \in R(G)$. Then the K-theoretic Euler class is defined by

$$e_G(V) = s^*(\tau_V),$$

where $s : * \to V$ is the zero map, i.e., $s(*) = 0 \in V$, and the induced map

$$s^* : K_G(V) = K_G(V^+, +) \to K_G(*) = K_G(* \coprod +, +).$$

is a ring homomorphism. It is well-known that

$$e_G(V) = \sum_{i=0}^{n}(-1)^i\Lambda^i(V) \in R(G),$$

where $\Lambda^i(V)$ is the i-th exterior representation of V and $n = \dim_{\mathbb{C}} V$, see [1], [3], [4], [5], [7].

Proposition 4.1. *The following hold.*

(1) $e_G(V \oplus W) = e_G(V)e_G(W)$ *in* $R(G)$.
(2) *If* $V^G \neq 0$, *then* $e_G(V) = 0$.

Proof. (1)

$$\begin{aligned} e_G(V \oplus W) &= \sum_{i=0}^{n} (-1)^i \Lambda^i(V \oplus W) \\ &= \sum_{i=0}^{n} \sum_{k+l=i} (-1)^{k+l} \Lambda^k(V) \otimes \Lambda^l(W) \\ &= e_G(V)e_G(W) \end{aligned}$$

(2) For the trivial representation $\mathbb{C}$, $e_G(\mathbb{C}) = \mathbb{C} - \Lambda(\mathbb{C}) = \mathbb{C} - \mathbb{C} = 0$. If $V^G \neq 0$, then V is decomposed as $V = V' \oplus \mathbb{C}$. By (2), $e_G(V) = e_G(V')e_G(\mathbb{C}) = 0$. □

We now compute the Euler class of every irreducible M_p-representation. The results are the following.

Theorem 4.2. *In the representation ring* $R(G)$ *of* $G = M_p$, *the following hold.*

(1) $e_G(V_{k,l}) = 1 - V_{k,l}$ $(k, l \in \mathbb{F}_p)$.
(2) $e_G(U_m) = \sum_{i=1}^{p-1} (-1)^i \frac{\binom{p}{i}}{p} U_{mi}$ $(m \in \mathbb{F}_p^*)$, *where* $\binom{p}{i}$ *denotes the binomial coefficient.*

Since a representation is detected by its character and the character is a class function on G, we have

Proposition 4.3. *Let* G *be a finite group and* $\mathcal{C}(G)$ *the set of representatives of conjugacy classes of cyclic subgroups of* G. *Then the restriction homomorphism*

$$\mathrm{Res} = (\mathrm{Res}^G_D) : R(G) \to \oplus_{D \in \mathcal{C}(G)} R(D)$$

is injective, see [10].

Proof of Theorem 4.2. (1) Since $V_{k,l}$ is 1-dimensional, $\Lambda^i V_{k,l} = 0$ for $i > 1$. By the definition of the Euler class, we obtain (1).

(2) Note that $\mathcal{C}(G)$ for $G = M_p$ consists of K_k for $0 \le k \le p$, C the center of M_p and $\{1\}$. Note also $\mathrm{Res}^G_{K_k} U_m \cong \mathbb{C}[C_p]$. Indeed,

$$\begin{aligned}\mathrm{Res}^G_{K_k} U_m &= \mathrm{Res}^G_{K_k} \mathrm{Ind}^G_{H_0} S_{0,m} \\ &= \mathrm{Ind}^{K_k}_{\{1\}} \mathrm{Res}^{H_0}_{\{1\}} S_{0,m} = \mathrm{Ind}^{K_k}_{\{1\}} \mathbb{C} \cong \mathbb{C}[C_p].\end{aligned}$$

Set $W = \displaystyle\sum_{i=1}^{p-1} (-1)^i \frac{\binom{p}{i}}{p} U_{mi}$. Then

$$\mathrm{Res}^G_{K_k} W \cong \sum_{i=1}^{p-1} (-1)^i \frac{\binom{p}{i}}{p} \mathbb{C}[C_p] \cong \frac{(1-1)^p}{p} \mathbb{C}[C_p] = 0.$$

On the other hand, we also see $\mathrm{Res}^G_{K_k} e_G(U_m) = e_G(\mathbb{C}[C_p]) = 0$, since $\mathbb{C}[C_p]^{C_p} = \mathbb{C}$. Therefore, we conclude that $\mathrm{Res}^G_{K_k} e_G(U_m) = \mathrm{Res}^G_{K_k} W$.

Since $\mathrm{Res}^G_C U_m = pT_m$, where T_m is the 1-dimensional irreducible representation of $C \cong C_p$ given by $cz = \xi_p^m$, it follows that

$$\begin{aligned}\mathrm{Res}^G_C W &\cong \sum_{i=1}^{p-1} (-1)^i \binom{p}{i} T_{mi} \cong \sum_{i=0}^{p} (-1)^i \binom{p}{i} T_m^{\otimes i} \\ &\cong (1 - T_m)^p \cong \mathrm{Res}^G_C e(U_m).\end{aligned}$$

Clearly $\mathrm{Res}^G_{\{1\}} e(U_m) = \mathrm{Res}^G_{\{1\}} W$ $(= 0)$. Thus we obtain (2) by Proposition 4.3. □

Finally we provide a couple of examples.

Example 4.4. If a G-representation V includes $V_{k,l} \oplus U_m$ for some k, l, m, then $e_G(V) = 0$.

Proof. It suffices to show $e_G(V_{k,l} \oplus U_m) = 0$ by Proposition 4.1. By Proposition 3.2 and Theorem 4.2, we see

$$e_G(V_{k,l} \oplus U_m) = e_G(V_{k,l}) e_G(U_m) = (1 - V_{k,l}) e_G(U_m) = e_G(U_m) - e_G(U_m) = 0.$$

□

Example 4.5. If a G-representation V includes

$$U := V_{0,p-1} \oplus V_{1,p-1} \oplus \cdots \oplus V_{p-1,p-1} \oplus V_{1.0},$$

then $e_G(V) = 0$.

Proof. It suffices to show $e_G(U) = 0$. By Theorem 4.2,

$$e_G(U) = (1 - x) \prod_{i=0}^{p-1} (1 - x^i y^{p-1}),$$

where $x = V_{1,0}$ and $y = V_{0,1}$. Note that $\mathrm{Res}^G_{K_k}(1 - x^k y^{p-1}) = 0$ for $0 \le k \le p-1$ and $\mathrm{Res}^G_{K_p}(1 - x) = 0$. Moreover, $\mathrm{Res}^G_C(1 - x) = 0$ and $\mathrm{Res}^G_{\{1\}}(1 - x) = 0$. Therefore we obtain that $\mathrm{Res}^G_D\, e_G(U) = 0$ for all $D \in \mathcal{C}(G)$. This implies $e_G(U) = 0$. □

Remark. These examples also follow from the following fact: If $V^D \neq 0$ for every $D \in \mathcal{C}(G)$, then $e_G(V) = 0$, see [7].

Conflict of Interest. The author has no conflicts of interest directly relevant to the content of this article.

References

[1] M. F. Atiyah and D. O. Tall, *Group representations, λ-rings and the J-homomorphism*, Topology **8** (1969), 253–297.
[2] T. Bartsch, *On the genus of representation spheres.*, Comment. Math. Helv. **65** (1990), 85-95.
[3] J.-H. Cho, *A survey on equivariant K-theory of representation spheres*, Trends in Mathematics Information Center for Mathematical Sciences **4** (2001), 84–89.
[4] T. tom Dieck, Transformation groups and representation theory, Lecture Notes in Mathematics **766**, Springer, Berlin, 1979.
[5] T. tom Dieck, Transformation groups, *Transformation Groups*, Walter de Gruyter, Berlin, New York, 1987
[6] D. Gorenstein, *Finite Groups*, Second edition, AMS Chelsea, 2007.
[7] K. Komiya, *Equivariant K-theoretic Euler classes and maps of representation spheres*, Osaka J. Math. 38 (2001), 239–249.
[8] I. Nagasaki, *Elementary abelian p-groups are the only finite groups with the Borsuk-Ulam property*, J. Fixed Point Theory Appl. **21** (2019), Article 16.
[9] K. Ohashi, *Equivariant maps between representation spheres of cyclic p-groups.* Osaka J. Math. **54** (2017), 647-659.
[10] J. P. Serre, Linear representations of finite groups. Graduate Texts in Mathematics, Vol. 42. Springer-Verlag, New York-Heidelberg, 1977.