Equivariant maps between C_{2p}-representation spheres for an odd prime p

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Abstract. In the previous research, we discussed a necessary and sufficient condition for the existence of a G-map between unitary representation spheres of a cyclic group C_{pq}, where p and q are distinct primes. In this paper, we study a similar problem for *orthogonal* representation spheres of C_{pq}. In particular, we treat the case of C_{2p}, where p is an odd prime. As a result, we show that some results in the unitary case do not hold in the orthogonal case.

1. Background

Let G be a finite group. In equivariant topology, the following question is fundamental and important for application to topological problems.

Question. Given G-representation spheres $S(V)$ and $S(W)$, does there exist a G-map $f : S(V) \to S(W)$ or not?

For example, a kind of non-existence result on an equivariant map plays a crucial role in the proof of Furuta's 10/8-theorem [3]. If the G-fixed point set $S(W)^G$ is not empty, then clearly a G-map always exists, and if $S(V)^G \neq \emptyset$ and $S(W)^G = \emptyset$, then there are no G-maps. Therefore we assume that representations V and W are *G-fixed-point-free*; i.e, $V^G = W^G = 0$ unless otherwise stated.

Equivariant obstruction theory provides several results on the above question, for example, see [5, 6, 9]. However a complete answer is not obtained at present since the computation of obstruction classes is difficult in general. In [7], we have treated unitary representations of a cyclic group C_{pq}, where p, q are distinct primes, and give an answer of the question as follows:

Proposition 1.1 ([7]). *Let $G = C_{pq}$ and V and W be unitary G-representations with $V^G = W^G = 0$. Then there exists a G-map $f\colon S(V) \to S(W)$ if and only if the following conditions hold.*

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(1) $\dim V^{C_p} \leq \dim W^{C_p}$ *and* $\dim V^{C_q} \leq \dim W^{C_q}$.
(2) *If* $\dim W^{C_p} = 0$ *or* $\dim W^{C_q} = 0$, *then* $\dim V \leq \dim W$.

In this paper, we consider *orthogonal* C_{pq}-representations. If p, q are odd primes, then the same result holds because any orthogonal representation of a finite group of odd order has a complex structure. On the other hand, if $q = 2$ and p is an odd prime, then Proposition 1.1 does not hold as mentioned in section 3. One of the purposes of this paper is to present such a counterexample.

2. Preliminary facts on C_n-maps

Let us recall irreducible orthogonal representations of a cyclic group C_n of order n. Let a be a generator of C_n. By Serre [8], irreducible unitary C_n-representations $U_k\,(= \mathbb{C})$, $k \in \mathbb{Z}/n$, are given by $az = \xi_n^k z$, $z \in U_k$, where $\xi_n = \exp(2\pi\sqrt{-1}/n)$. We set $T_k = \mathrm{res}_{\mathbb{R}}\, U_k$. Then (2-dimensional) irreducible orthogonal C_n-representations are given by T_k when $k \not\equiv 0 \pmod n$ or $k \not\equiv n/2 \pmod n$ if n is even. If n is even and $k = n/2$, there is a 1-dimensional representation $\mathbb{R}_\varepsilon$, where $\varepsilon : C_n \to \{\pm 1\}$ is the sign homomorphism. The action of C_n on $\mathbb{R}_\varepsilon$ is given by $gx = \varepsilon(g)x$ for any $g \in C_n$ and $x \in \mathbb{R}_\varepsilon$; in particular, $ax = -x$. Not that $T_k \cong T_l$ as orthogonal representations if $kl \equiv 1 \bmod n$. Note $T_0 = 2\mathbb{R}$ and $T_{n/2} = 2\mathbb{R}_\varepsilon$ if n is even. Summarizing these facts, we have

Proposition 2.1. *All orthogonal irreducible representations of* C_n *are given as follows.*

(1) *When n is odd, there are $(n-1)/2$ 2-dimensional irreducible representations T_k $(1 \leq k \leq (n-1)/2)$ and there is a 1-dimensional trivial representation $\mathbb{R}$.*
(2) *When n is even, there are $(n-2)/2$ 2-dimensional irreducible representations T_k $(1 \leq k \leq n/2 - 1)$ and there are 1-dimensional representations $\mathbb{R}$ and $\mathbb{R}_\varepsilon$.*

We next discuss the existence of a G-map between $S(U_k)$ and $S(U_l)$, where k, $l \not\equiv 0 \bmod n$. Some special cases are described in [7].

Proposition 2.2. *Let $G = C_n$. There exists a G-map $f : S(U_k) \to S(U_l)$ if and only if (k,n) divides (l,n), where (k,n) denotes the greatest common divisor of k and n.*

Proof. Set $d = (k,n)$ and $e = (l,n)$. If there exists a G-map $f : S(U_k) \to S(U_l)$, then for any $x \in S(U_k)$, it follows that

$$G_x = \mathrm{Ker}\, U_k = \langle a^{n/d} \rangle \cong C_d \leq G_{f(x)} = \mathrm{Ker}\, U_l = \langle a^{n/e} \rangle \cong C_e.$$

Hence d divides e.

We show the converse. Assume that $d = (k, l)$ divides $e = (l, n)$. Recall that the action of the generator $a \in G$ on U_k is given by $az = \xi_n^k z$, $z \in U_k$. Therefore one see that $K := \operatorname{Ker} U_k = C_d \leq G$. Furthermore U_k is regarded as G/K-representation and $G/K \cong C_{n/d}$ acts freely on U_k. By assumption, it follows that $L := \operatorname{Ker} U_l \geq K$ and G/K acts on U_l with the kernel L/K. If there exists a G/K-map, then one can obtain a G-map by inflation of the G/K-map. Therefore one may assume that $(k, n) = 1$. Take an integer k' with $kk' \equiv 1 \bmod n$ and define a map $f : S(U_k) \to S(U_l)$ by $f(z) = z^{k'l}$ for $z \in U_k$. One can easily see that this map is G-equivariant. □

Remark. If we set $(0, n) = n$, then the above proposition holds for $k = 0$ or $l = 0$.

By restricting the ground field $\mathbb{C}$ to $\mathbb{R}$, we obtain

Corollary 2.3. *There exists a G-map $f : S(T_k) \to S(T_l)$ if and only if (k, n) divides (l, n).*

3. The case of C_{2p}

In this section, G is a cyclic group C_{2p} of order $2p$, where p is an odd prime. Let V and W be orthogonal G-representations with $V^G = W^G = 0$. We consider the question whether a G-map from $S(V)$ to $S(W)$ exists. The non-trivial irreducible representations are:

$$T_i \ (1 \leq i \leq p-1), \quad \mathbb{R}_\varepsilon.$$

Note that $\operatorname{Ker} T_i = 1$ for odd i and $\operatorname{Ker} T_i = C_2$ for even i. By the same argument of [7], we obtain the following fact.

Proposition 3.1. *If there exists a G-map $f : S(V) \to S(W)$, then*

(C1) $\dim V^{C_p} \leq \dim W^{C_p}$ *and* $\dim V^{C_2} \leq \dim W^{C_2}$.

(C2) *If* $\dim W^{C_p} = 0$ *or* $\dim W^{C_2} = 0$, *then* $\dim V \leq \dim W$.

We would like to consider the converse. We first note that

Theorem 3.2. *In addition to* (C1) *and* (C2), *if* $\dim W^{C_p} \geq 2$ *is satisfied, then there exists a G-map $f : S(V) \to S(W)$.*

Proof. Since the proof is similar with [7], we only give an outline. Let

$$V = a_1 T_1 \oplus \cdots \oplus a_{p-1} T_{p-1} \oplus c\mathbb{R}_\varepsilon \quad (a_i \geq 0,\ c \geq 0)$$

be the irreducible decomposition of V. For any subgroup K of G, let $V(K)$ denote the direct sum of irreducible representations with kernel K in the irreducible decomposition of V. So we have

$$V(1) = \bigoplus_{i:\text{odd}} a_i T_i \quad V(C_2) = \bigoplus_{i:\text{even}} a_i U_i \quad V(C_p) = c\mathbb{R}_\varepsilon.$$

Hence $V = V(1) \oplus V(C_2) \oplus V(C_p)$. Similarly we obtain $W = W(1) \oplus W(C_2) \oplus W(C_p)$. Note also that $V^{C_2} = V(C_2)$ and $V^{C_p} = V(C_p)$. If $\dim W(C_2) = 0$, then $V(C_2) = 0$ and one can easily see the existence of a G-map from condition (C2). Assume $\dim W(C_2) > 0$; in fact $\dim W(C_2) \geq 2$. Let C be C_p or C_2. Since $W^C = W(C)$, it follows that $\dim W^C \geq 2$ by assumption. Then one can find a G-map $h : S(W) \to S(W)$ with $\deg h = 0$ using equivariant obstruction theory.

One can construct a G-map $f^{>1} : S(V)^{>1} \to S(W)$, where

$$S(V)^{>1} = S(V)^{C_p} \coprod S(V)^{C_2}$$

is the singular set of $S(V)$. By composing $f^{>1}$ with h, it follows that the equivariant obstruction to the extension of $h \circ f^{>1}$ vanishes. Hence there exists a G-map $f : S(V) \to S(W)$. □

From the above proposition, the remaining case is

$$\dim W^{C_p} = 1 \text{ and } \dim W^{C_2} > 0.$$

Proposition 3.3. *In this case, for any G-map $h : S(W) \to S(W)$, it follows that* $\deg h^{C_p} = \pm 1$ *and* $\deg h \equiv \pm 1 \bmod p$. *In particular* $\deg h \neq 0$.

Proof. Note that $\deg h^G = 1$ since $S(W)^G = \emptyset$. Since

$$h^{C_p} : S(W)^{C_p} = S^0 \to S(W)^{C_p} = S^0$$

is $G/C_p \cong C_2$-map and C_2 acts freely on $S(W)^{C_p}$, it follows that $\deg h^{C_p} = \pm 1$. By the Burnside relation, see [1, 2], we obtain $\deg h^{C_2} \equiv 1 \bmod p$ and

$$\deg h + \deg h^{C_2} + (p-1) \deg h^{C_p} + (p-1) \deg h^G \equiv 0 \bmod 2p.$$

Reducing this to $\bmod p$, we have

$$\begin{aligned} \deg h &\equiv -\deg h^{C_2} + \deg h^{C_p} + \deg h^G \bmod p \\ &\equiv \deg h^{C_p} \equiv \pm 1 \bmod p. \end{aligned}$$

□

This result means that the argument in Theorem 3.2 is not available; namely, we cannot prove the vanishing of the obstruction.

As an easy corollary, we obtain a variation of Borsuk-Ulam results, cf. [10].

Corollary 3.4. *In the above situation, If $W \subsetneq U$, then there are no G-maps from $S(U)$ to $S(W)$.*

Proof. If there exists a G-map $f : S(U) \to S(W)$, then $h := i \circ f : S(W) \to S(W)$ has a non-zero degree, where i is the inclusion. On the other hand, by dimensional reason, $\deg h = 0$; this is a contradiction. □

Under the condition $\dim V^{C_2} \leq \dim W^{C_2}$, we next discuss the question in the following two cases:

(1) $\dim V^{C_p} = \dim W^{C_p} = 1$.

(2) $\dim V^{C_p} = 0$ and $\dim W^{C_p} = 1$.

we here provide two examples. The first example shows that Proposition 1.1 does not hold in orthogonal case. Let $G = C_{2p}$ as before.

Example 3.5. Let $V = T_1 \oplus T_1 \oplus \mathbb{R}_\varepsilon$ and $W = T_2 \oplus \mathbb{R}_\varepsilon$. There are no G-maps from $S(V)$ to $S(W)$.

Proof. Suppose that there exists a G-map $f : S(V) \to S(W)$. Set $U = T_1 \oplus \mathbb{R}_\varepsilon$. Consider a C_p-map $h := \mathrm{res}_{C_p} f_{|S(U)} : S(U) \to S(W)$. Then $\deg h \equiv \pm 2 \bmod p$. In fact, $h^{C_p} = \pm id$ and there is a G-map $k : S(U) \to S(W)$ with $\deg k = \pm 2$ and $k^{C_p} = h^{C_p}$. By a result of equivariant obstruction theory [1, 2], for any G-map $h : S(U) \to S(W)$, it follows that $\deg h - \deg k \equiv 0 \bmod p$. As a result, we have $\deg h \neq 0$.

On the other hand, there exists a 3-disk $D^3 \subset S(V)$ such that $\partial D^3 = S(U)$, and so $\deg h = 0$; this is a contradiction. In fact, D^3 can be taken as follows. Note that

$$\begin{aligned} S(V) &\cong S(T_1) \times D(U) \cup D(T_1) \times S(U) \subset \mathbb{R}^2 \times \mathbb{R}^3 \\ &= \{(x, y) \mid \|x\| = 1, \|y\| \leq 1\} \cup \{\{(x, y) \mid \|x\| \leq 1, \|y\| = 1\}, \end{aligned}$$

and $S(U)$ is regarded as $0 \times S(U) = \{(0, y) \mid \|y\| = 1\}$. Consider the following sets:

$$D = \{(1, y) \mid \|y\| \leq 1\} \cong D^3 \text{ and } C = \{(t, y), \mid 0 \leq t \leq 1, \|y\| = 1\} \cong I \times S(U).$$

By attaching C to D on $\partial D = 1 \times S(U)$, we obtain $X = C \cup_{1 \times S(U)} D$. Then $X \cong D^3$ and $\partial X = S(U)$. □

Remark. By [4], h is equivariantly desuspended to a map $h' : S(T_1) \to S(T_2)$ as C_p-maps. This fact also shows that $\deg h \neq 0$.

The next example is a counterexample to the Borsuk-Ulam theorem.

Example 3.6. Let $V = T_1 \oplus T_2$ and $W = T_2 \oplus \mathbb{R}_\varepsilon$. There exists a G-map $f : S(V) \to S(W)$.

Proof. Note that

$$S(V) \cong S(T_1) * S(T_2) = D(T_1) \times S(T_2) \cup S(T_1) \times D(T_2).$$

Take a G-map

$$\bar{h} : S(T_2) = 0 \times S(T_2) \ (\subset S(V)) \to S(T_2) = S(T_2) \times 0 \ (\subset S(W))$$

with even degree. For example, one can take a G-map defined by $\bar{h}(z) = z^{p+1}$ whose degree $\deg \bar{h} = p + 1$ is even.

In the same way as the proof of Example 3.5, we can take a 2-disk $D^2 \subset S(V)$ whose boundary is $S(T_2) = 0 \times S(T_2) \cong S^1$. Set $S_2 := D^2 \cup aD^2 \subset S(V)$ which is homeomorphic to a 2-sphere S^2, where a is a generator of G.

Since $\bar{h}$ is of even degree, we can construct a (non-equivariant) map $h : S_2 \to S(W)$ extending $\bar{h}$ such that $\deg h = 0$. In fact, we can change the degree of h by any even number.

Finally, take a 3-disk $D^3 \subset S(V)$ which is a region between D^2 and aD^2; hence S_2 is the boundary of D^3. Then it follows that

$$S(V) = \bigcup_{g \in G} (gD^3).$$

Since $\deg h = 0$, there exists a (non-equivariant) map $\widetilde{h} : D^3 \to S(W)$ extending h. Using this, we can define a G-map

$$f := \bigcup_{g \in G} (g \cdot \widetilde{h}) : S(V) = \bigcup_{g \in G} (gD^3) \to S(W)$$

by $f(x) = g\widetilde{h}(g^{-1}x)$ for any $x \in gD^3$. One can easily check that this map is a well-defined G-map. □

Corollary 3.7. *There exists a G-map $g : S(T_1 \oplus T_1) \to T_2 \oplus \mathbb{R}_\varepsilon$.*

Proof. By Proposition 2.2, there is a G-map $k : S(T_1) \to S(T_2)$. This G-map k induces a G-map $h = id * k : S(T_1 \oplus T_1) \to S(T_1 \oplus T_2)$. Composing h with f, we obtain a G-map $g = f \circ h$. □

Remark. There are no G-maps from $S(T_1 \oplus T_1)$ to $S(T_1 \oplus \mathbb{R}_\varepsilon)$ because the condition (C2) of Proposition 3.1 is not satisfied.

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