Estimation of the isovariant Borsuk-Ulam constant of SO(3)

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Abstract. Let SO(3) be the the special orthogonal group in dimension 3. We shall compute the fixed point dimensions of the irreducible SO(3)-representations. As an application, we shall provide a new estimate of the isovariant Borsuk-Ulam constant of SO(3).

1. **Introduction**

Let G be a compact Lie group and V, W orthogonal G-representations. A G-map $f : V \to W$ is called G-isovariant map if $G_x = G_{f(x)}$ for all $x \in V$, where G_x denotes the isotropy subgroup of $x \in V$. In other words, a G-isovariant map f is a G-map being injective on each orbit $G(x)$ of x.

A fundamental question is the following:

Question. When does there exist a G-isovariant map between orthogonal G-representations?

Concerning this question, Wasserman [11] shows the isovariant Borsuk-Ulam theorem when G is solvable. Namely, for any solvable compact Lie group, if there exists a G-isovariant map $f : V \to W$, then

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds. We say that G is a *Borsuk-Ulam group* if the isovariant Borsuk-Ulam theorem holds for G. Several Borsuk-Ulam groups are known. For example, the following are Borsuk-Ulam groups.

- Solvable groups (Wasserman [11]).
- A_n, S_n for $n \leq 27$ (Wasserman [11] for $n \leq 11$, Sumi [9] and [10] for $12 \leq n \leq 27$).
- PSL$(2, q)$ (Nagasaki-Ushitaki [7]), PSL$(3, q)$, PSU$(3, q)$ (Sumi [10]).

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- All sporadic finite simple groups (Sumi [9]).

However a complete solution is not known and there are no examples of connected compact Lie groups other than n-tori so far. We conjecture that SO(3) is a Borsuk-Ulam group, but there is no proof at present.

In order to approach this conjecture, we recall the isovariant Borsuk-Ulam constant c_G which was first introduced in [5].

Definition ([5])**.** The number c_G is the supremum of $c \in \mathbb{R}$ with the following property:

$$c\,(\dim V - \dim V^G) \leq \dim W - \dim W^G$$

holds for any pair (V, W) with a G-isovariant map $f : V \to W$.

Clearly, $0 \leq c_G \leq 1$ and G is a Borsuk-Ulam group if and only if $c_G = 1$. We provided some estimates of c_G for connected compact Lie groups in the previous papers [3], [4], [5] and [6]. In particular, we showed $c_{\mathrm{SO}(3)} \geq 4/5$. In this article, we shall give a better estimate of $c_{\mathrm{SO}(3)}$. The main result is

Theorem 1.1. $c_{\mathrm{SO}(3)} \geq 12/13$.

2. A Key result

First we remark that it is sufficient that representations in consideration are unitary as discussed in [5]. Therefore, in the following, we assume that G-representations are unitary. Let H be a closed subgroup of a compact Lie group G. We set

$$d_{(G,H)} = \sup_U \frac{\dim U^H}{\dim U}$$

where U is taken over nontrivial irreducible G-representations. In order to estimate c_G, we use the following result.

Proposition 2.1 ([4], [5])**.** *If H is a Borsuk-Ulam group, then*

$$c_G \geq 1 - d_{(G,H)}.$$

Let $\mathcal{B}$ be a family of conjugacy classes (H) of closed subgroups of G with $c_H = 1$. Set

$$d(\mathcal{B}) = \inf_{(H)\in\mathcal{B}} d_{(G,H)}.$$

If closed subgroups H and K are conjugate in G, then $\dim U^H = \dim U^K$ and $d(G,H) = d(G,K)$. Therefore $d(\mathcal{B})$ is well-defined. By Proposition 2.1, we have

Corollary 2.2. *With the above notation,* $c_G \geq 1 - d(\mathcal{B})$.

Let G be SO(3). Then the representatives of conjugacy classes of proper closed subgroups are given by the following list, see [2], [12].

- C_n: cyclic group of order $n \geq 1$.
- D_n: dihedral group of order $2n \geq 2$.
- $T \cong A_4$: tetrahedral group.
- $O \cong S_4$: octahedral group.
- $I \cong A_5$: icosahedral group.
- $\mathrm{SO}(2) \cong S^1$: special orthogonal group, which is a maximal torus.
- O(2): orthogonal group, which is the normalizer of SO(2).

Note that every proper closed subgroup of SO(3) is a Borsuk-Ulam group. In fact, $I \cong A_5$ satisfies the prime condition in [11], and so I is a Borsuk-Ulam group. The other groups are solvable groups, hence, Borsuk-Ulam groups. Let $\mathcal{B}$ be the set of conjugacy classes of proper closed subgroups of SO(3) and $\mathcal{M}$ the set of maximal conjugacy classes in $\mathcal{B}$. It is easily seen that

$$\mathcal{M} = \{(O),\ (I),\ (\mathrm{O}(2))\}.$$

Lemma 2.3. *With the above notation,*

(1) $d(\mathcal{B}) = d(\mathcal{M})$.

(2) $c_{\mathrm{SO}(3)} \geq 1 - d(\mathcal{M})$.

Proof. (1) For any $(H) \in \mathcal{B}$, there exists an element (K) in $\mathcal{M}$ such that $(K) \geq (H)$, where $K = O$, I or O(2). Therefore, $\dim U^H \geq \dim U^K$ and so $d_{(G,H)} \geq d_{(G,K)}$. This implies that $d(\mathcal{B}) \geq d(\mathcal{M})$. On the other hand, since $\mathcal{B} \supset \mathcal{M}$, it follows from definition that $d(\mathcal{B}) \leq d(\mathcal{M})$. Thus $d(\mathcal{B}) = d(\mathcal{M})$.

(2) It is straightforward from Corollary 2.2. □

3. Fixed point dimensions

Let Γ be a finite group and V a unitary Γ-representation. We first recall a formula on $\dim V^\Gamma$. We use the following notations.

- $S(\Gamma)$: the set of subgroup of Γ.
- $cS(\Gamma)$: the set of conjugacy classes of subgroups in Γ.
- $Cy(\Gamma)$: the set of cyclic subgroups of Γ.
- $cCy(\Gamma)$: the set of conjugacy classes of cyclic subgroups in Γ.

Let H be a subgroup of Γ. By [8], $\dim V^H = \dfrac{1}{|H|}\sum_{g\in H}\chi_V(g)$, where χ_V is the character of V. Set $\dim(H) = \dim V^H$ and $f(H) = \sum_{g\in H}\chi_V(g) = |H|\dim(H)$. Then

$$f(H) = \sum_{C\in Cy(H)}\sum_{g\in C^*}\chi_V(g),$$

where C^* is the set of generators of C. Set $g(C) = \sum_{g\in C^*}\chi_V(g)$. Using a similar argument as in [7], we obtain

$$(*):\quad f(H) = \sum_{C\in Cy(H)}\Big(\sum_{C\le D\in Cy(H)}\mu(C,D)\Big)f(C),$$

where μ is the Möbius function on the subgroup lattice $S(H)$. Set

$$m_H(C) = \sum_{C\le D\in Cy(H)}\mu(C,D).$$

Clearly, if C, $C'\in Cy(H)$ are conjugate in H, then $m_H(C) = m_H(C')$. The number of cyclic subgroups $C'\in Cy(H)$ conjugate to C in H is equal to $|H|/|N_H(C)|$, where $N_H(C)$ is the normalizer of C in H. Thus we obtain the following formula.

Proposition 3.1.

$$\dim(H) = \sum_{(C)\in cCy(H)}\frac{|C|}{|N_H(C)|}m_H(C)\dim(C).$$

Proof. Since $f(H) = |H|\dim(H)$ and $f(C) = |C|\dim(C)$, by equation $(*)$, we have

$$\begin{aligned}\dim(H) &= \sum_{C\in Cy(H)}\frac{|C|}{|H|}m_H(C)\dim(C)\\ &= \sum_{(C)\in cCy(H)}\frac{|H|}{|N_H(C)|}\frac{|C|}{|H|}m_H(C)\dim(C)\\ &= \sum_{(C)\in cCy(H)}\frac{|C|}{|N_H(C)|}m_H(C)\dim(C).\end{aligned}$$

□

We here consider the cases of the octahedral group O and the icosahedral group I, which are isomorphic to S_4 and A_5 respectively. In these cases, the dimensions of fixed point sets are described in [2, p.260], but those results are somewhat unclear. So we provide the details in the following.

In the case of the octahedral group $O \cong S_4$, it is easily seen that S_4 has cyclic subgroups of order 1, 2, 3 and 4. The representatives of conjugacy classes in $cCy(S_4)$

are are given by the following table. Furthermore, $N_{S_4}(C_n) = D_n$ for $n = 2$, 3, 4 and

	1	C_2	C_2'	C_3	C_4
generator	1	$(1,2)$	$(1,3)(2,4)$	$(1,2,3)$	$(1,2,3,4)$
order	1	2	2	3	4

$N_{S_4}(C_2') = D_4$. Note that C_2, C_3 and C_4 are maximal cyclic subgroups in S_4. Hence $m(C_2) = m(C_3) = m(C_4) = 1$. The cyclic subgroups including C_2' are C_2' itself and one isomorphic to C_4. Therefore $m(C_2') = 0$. The numbers of cyclic subgroups of order n are given by the following table.

$\lvert C\rvert$	1	2	3	4
#	1	9	4	3
$\mu(1,C)$	1	-1	-1	0

Thus it follows that $m(1) = -12$. Therefore, we obtain the following from Proposition 3.1.

Proposition 3.2.

$$\dim(S_4) = \frac{1}{2}(-\dim(1) + \dim(C_2) + \dim(C_3) + \dim(C_4)).$$

Next, we consider the case of $I \cong A_5$. The argument is similar. A_5 has cyclic subgroups C of order 1, 2, 3 and 5. The representatives of conjugacy classes in $cCy(A_5)$ are given buy the following table.

	1	C_2	C_3	C_5
generator	1	$(1,3)(2,4)$	$(1,2,3)$	$(1,2,3,4,5)$
order	1	2	3	5

Furthermore, $N_{A_5}(C_n) = D_n$ for $n = 2$, 3, 5. Note that C_2, C_3 and C_5 are maximal cyclic subgroups in A_5. Hence $m(C_2) = m(C_3) = m(C_5) = 1$.

The numbers of cyclic subgroups of order n are given by the following table.

$\lvert C\rvert$	1	2	3	5
#	1	15	10	6
$\mu(1,C)$	1	-1	-1	-1

Thus it follows that $m(1) = -30$. Therefore, we obtain the following from Proposition 3.1.

Proposition 3.3.

$$\dim(A_5) = \frac{1}{2}(-\dim(1) + \dim(C_2) + \dim(C_3) + \dim(C_5)).$$

4. Proof of Theorem 1.1

Let us recall the irreducible representations of $G = \mathrm{SO}(3)$. For each $n \geq 0$, there exists only one G-representation U_n of dimension $2n+1$ and the character χ_n on a maximal torus S^1 is given by

$$\chi_n(t) = 1 + (t + t^{-1}) + \cdots + (t^n + t^{-n})$$

for $t \in S^1 = \{t \in \mathbb{C} \mid |t| = 1\}$, see [1].

We shall compute $\dim U_n^H$ for $(H) \in \mathcal{M}$. In the case of $H = \mathrm{O}(2)$, we know

$$\dim U_n^{\mathrm{O}(2)} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even,} \end{cases}$$

which was shown in [3]. Thus we obtain

Proposition 4.1 ([3])**.** $d_{(G,\mathrm{O}(2))} = 1/5$.

Let C be a finite cyclic group of G. Since maximal tori are conjugate each other, we may suppose that C is a subgroup of a fixed maximal torus S^1. Then we have

Proposition 4.2. *Let $C = C_m$ be a cyclic subgroup of order m. Then*

$$\dim U_n^{C_m} = 1 + 2\left[\frac{n}{m}\right],$$

where $[x]$ denotes the greatest integer not exceeding x.

Proof. It is seen that

$$\mathrm{Res}_{C_m} U_n = \mathbb{C} \oplus \oplus_{i \geq 1}(T_i \oplus \overline{T_i}),$$

where T_i is the 1-dimensional representation given by $\xi_m \cdot z = \xi_m^i z$ $(z \in T_i)$ and $\overline{T_i}$ is the complex conjugate of T_i. Then

$$\dim(T_i \oplus \overline{T_i})^{C_m} = \begin{cases} 2 & m \mid i \\ 0 & \text{otherwise.} \end{cases}$$

This shows the desired result. □

Proposition 4.3 (cf. [2, p.260])**.** *The following hold.*

(1)

$$\dim U_n^O = \begin{cases} [n/12] & n \equiv 1, 2, 3, 5, 7, 11 \ (12) \\ 1 + [n/12] & n \equiv 0, 4, 6, 8, 9, 10 \ (12), \end{cases}$$

(2)

$$\dim U_n^I = \begin{cases} [n/30] & n \equiv 1,2,3,4,5,7,8,9,11,13,14,17,19,23,29\ (30) \\ 1+[n/30] & n \equiv 0,6,10,12,15,16,18,20,21,22,24,25,26,27,28\ (30). \end{cases}$$

Proof. Using Propositions 3.2, 3.3 and 4.2, we obtain
(1) $\dim U_n^O = 1 - n + [n/2] + [n/3] + [n/4]$, and
(2) $\dim U_n^I = 1 - n + [n/2] + [n/3] + [n/5]$.
One can easily check that these coincide with the desired results. □

By Proposition 4.3, the following is straightforward.

Proposition 4.4. *The following hold.*
(1) $d_{(G,O)} = 1/9$.
(2) $d_{(G,I)} = 1/13$.

Propositions 4.1 and 4.4 directly show that $d_G = 1/13$. Therefore we obtain that $c_G \geq 12/13$. Thus the proof is completed.

Corollary 4.5. *For the unitary group* SU(2)*, it holds that* $c_{\mathrm{SU}(2)} \geq 12/13$.

Proof. By a result of [5], it follows that $c_{\mathrm{SU}(2)} = c_{\mathrm{SO}(3)} \geq 12/13$. □

Conflict of Interest. The author has no conflicts of interest directly relevant to the content of this article.

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