The Borsuk-Ulam Inequality For Representations

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Abstract. For any (virtual) representation $\alpha = W - V \in RO(G)$ of a finite group G, an integer-valued function d is defined by $d(H) = \dim \alpha - \dim \alpha^H$ for $H \leq G$. In this paper, we shall investigate a question whether the Borsuk-Ulam inequality $d(G) \geq 0$ holds when $d(C) \geq 0$ for every cyclic subgroup C of G, raised by research of the isovariant Borsuk-Ulam theorem, and we then determine finite abelian groups having such property for every $\mathcal{C}_G$-pair and finally provide a variant of the Borsuk-Ulam theorem.

1. Introduction

By Wasserman's work [6], we know that the isovariant Borsuk-Ulam theorem holds for a finite solvable group G; namely, if there is a G-isovariant map $f : V \to W$ between G-representations, then the *Borsuk-Ulam inequality*

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

or equivalently

$$\dim \alpha - \dim \alpha^G \geq 0 \quad (\alpha = W - V \in RO(G))$$

holds for solvable G.

Let $S(G)$ be the set of subgroups of G. For a given pair (V, W) of G-representations, we define an integer-valued function d on $S(G)$ by

$$\begin{aligned} d(H) &= \dim W - \dim W^H - \dim V + \dim V^H \\ &= \dim \alpha - \dim \alpha^H \quad (\alpha = W - V \in RO(G), H \in S(G)). \end{aligned}$$

Let $\mathcal{F}$ be a family of subgroups of G. We call (V, W) an $\mathcal{F}$-pair if $d(H) \geq 0$ for every $H \in \mathcal{F}$. We here consider the family of cyclic subgroups of G, denoted by $\mathcal{C}_G$. We also set $\mathcal{C}_G^0 = \mathcal{C}_G \setminus \{1\}$, where 1 denotes the unit element of G. In this paper, we shall investigate a question whether the Borsuk-Ulam inequality $d(G) \geq 0$ holds for every $\mathcal{C}_G$-pair. One of the main results is the following.

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Theorem 1.1. *Let G be a finite abelian group. The Borsuk-Ulam inequality*

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds for every $\mathcal{C}_G$-pair (V, W) if and only if G is a cyclic group C_n or an elementary abelian p-group $(C_p)^k$.

The proof of the theorem is given in sections 2 and 3. In section 4, we shall provide several nonabelian examples that the Borsuk-Ulam inequality holds for every $\mathcal{C}_G$-pair and we also show a variant of the Borsuk-Ulam theorem:

Theorem 1.2. *Let G be a finite group consisting of prime order elements and the unit element. For G-representations V and W, if, for each $C \in \mathcal{C}_G$, there is a C-map $f_C : S(V) \smallsetminus S(V)^C \to S(W) \smallsetminus S(W)^C$, then the Borsuk-Ulam inequality*

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds.

Here if $V = V^C$, then f_C is understood to be a C-map on the empty set. The finite groups satisfying the assumption in Theorem 1.2 are classified by [2]. An elementary abelian p-group $(C_p)^k$, a metacyclic group $Z_{p,q}$ of order pq, where p, q are primes and $q \mid p-1$, and the alternating groups A_4, A_5 are examples of such groups.

2. Algebraic description of the Borsuk-Ulam inequality

As mentioned in [6], the Borsuk-Ulam inequality is described by characters of representations. Let χ_α be a (virtual) character of $\alpha = W - V \in RO(G)$. Then $\dim \alpha = \chi_\alpha(1)$ and

$$\dim \alpha^H = \frac{1}{|H|} \sum_{g \in H} \chi_\alpha(g)$$

for a subgroup H of G. We define a function h by

$$h(H) = |H| d(H) = \sum_{g \in H} (\chi_\alpha(1) - \chi_\alpha(g)),$$

and a function k by

$$k(C) = \sum_{g \in C^*} (\chi_\alpha(1) - \chi_\alpha(g))$$

for any cyclic subgroup C, where C^* is the set of generators of C. Then h is described as

$$h(H) = \sum_{C \in \mathcal{C}_H} k(C).$$

In particular,

$$h(C) = \sum_{D\in\mathcal{C}_C} k(D)$$

for any $C \in \mathcal{C}_G$. Using the Möbius inversion on the subgroup lattice (see [1]), we obtain

$$k(D) = \sum_{C\in\mathcal{C}_D} \mu(C, D)h(C).$$

Therefore the following formula is obtained as proved in [5].

Proposition 2.1 ([5])**.**

$$\begin{aligned} h(G) &= \sum_{D\in\mathcal{C}_G}\sum_{C\in\mathcal{C}_D} \mu(C, D)h(C) \\ &= \sum_{C\in\mathcal{C}_G}\left(\sum_{C\le D\in\mathcal{C}_G} \mu(C, D)\right) h(C). \end{aligned}$$

Note that $k(1) = h(1) = 0$. By setting

$$m(C) = \sum_{C\le D\in\mathcal{C}_G} \mu(C, D)$$

for $C \in \mathcal{C}_G$, $h(G)$ is described as

$$h(G) = \sum_{C\in\mathcal{C}_G^0} m(C)h(C).$$

Since $h(H) = |H|d(H)$ by definition, we obtain

Corollary 2.2.

$$d(G) = \sum_{C\in\mathcal{C}_G^0} \frac{|C|}{|G|} m(C)d(C).$$

We now prove a half of Theorem 1.1.

Proposition 2.3. *If G is a cyclic group C_n or an elementary abelian p-group $(C_p)^k$, then the Borsuk-Ulam inequality holds for every $\mathcal{C}_G$-pair.*

Proof. When $G = C_n$, this is trivial by the definition of a $\mathcal{C}_G$-pair. We next consider the case of $G = (C_p)^k$. Suppose that $\alpha = W - V$ and (V, W) is a $\mathcal{C}_G$-pair; namely, $h(C) \ge 0$ holds for every $C \in \mathcal{C}_G$. For any nontrivial cyclic subgroup C of G, a cyclic subgroup D containing C is only C itself and therefore $m(C) = \mu(C, C) = 1$. Thus $h(G) = \sum_{C\in\mathcal{C}_G^0} h(C) \ge 0$ and this implies that $d(G) = \dim\alpha - \dim\alpha^G \ge 0$. □

3. **Proof of Theorem 1.1**

In this section, we prove another half of Theorem 1.1 after preparing some lemmas.

Let Q be a quotient subgroup of G and let $\pi : G \to Q$ be the projection. Through the projection π, any Q-representation V is thought of as a G-representation, which is called the *inflation* of V and denoted by $\widetilde{V}$ or $\mathrm{Inf}_Q^G V$.

Lemma 3.1. *If (V, W) is a $\mathcal{C}_Q$-pair, then $(\widetilde{V}, \widetilde{W})$ is a $\mathcal{C}_G$-pair.*

Proof. For any cyclic group C of G, set $\overline{C} = \pi(C) \in \mathcal{C}_Q$. For $\alpha = W - V$ and $\widetilde{\alpha} = \widetilde{W} - \widetilde{V}$, it follows that $\dim \widetilde{\alpha} - \dim \widetilde{\alpha}^C = \dim \alpha - \alpha^{\overline{C}} \geq 0$. This shows that $(\widetilde{V}, \widetilde{W})$ is a $\mathcal{C}_G$-pair. □

Lemma 3.2. *If the Borsuk-Ulam inequality holds for every $\mathcal{C}_G$-pair, then it holds for every $\mathcal{C}_Q$-pair.*

Proof. For any $\mathcal{C}_Q$-pair (V, W), the inflated pair $(\widetilde{V}, \widetilde{W})$ is a $\mathcal{C}_G$-pair by Lemma 3.1. Therefore $\dim \widetilde{\alpha} - \dim \widetilde{\alpha}^G \geq 0$ by assumption. Thus

$$d(Q) = \dim \alpha - \dim \alpha^Q = \dim \widetilde{\alpha} - \dim \widetilde{\alpha}^G \geq 0.$$

This means that the Borsuk-Ulam inequality holds for (V, W). □

If an abelian group G is neither cyclic nor elementary abelian, then there exists a subgroup H such that $G/H \cong (C_p)^2 \times C_q$, where p, q are distinct primes, or $G/H \cong C_p \times C_{p^2}$. By Lemma 3.2, the problem is reduced to the cases of $(C_p)^2 \times C_q$ and $C_p \times C_{p^2}$.

Let us first recall representations of a finite abelian group. Taking a subgroup K of G such that G/K is cyclic, one can obtain a complex 1-dimensional representation U_H with kernel K. Indeed, U_H is constructed as follows. Take a G/K-representation $U = \mathbb{C}$ on which a generator g of G/K acts by $gz = \zeta z$, where $z \in \mathbb{C}$ and ζ is a $|G/K|$-th primitive root of unity. Then U_K may be taken as the inflation $\widetilde{U}$ of U. The following is straightforward.

Lemma 3.3. $\dim U_K^H = \begin{cases} 2 & \textit{if} \quad H \leq K \\ 0 & \textit{if} \quad H \not\leq K. \end{cases}$

3.1. **The case of $G = (C_p)^2 \times C_q$.**

There are $p+1$ subgroups H_i $(0 \leq i \leq p)$ of $(C_p)^2$ such that $(C_p)^2/H_i \cong C_p$. Note also that $G/H_i \cong C_{pq}$ and

$$\mathcal{C}_G^0 = \{H_i, C_q, H_i \times C_q \mid 0 \leq i \leq p\}.$$

Consider G-representations:

$$V = U_{H_0\times C_q} \oplus \cdots \oplus U_{H_p\times C_q} \oplus U_{(C_p)^2},$$
$$W = U_{H_0} \oplus \cdots \oplus U_{H_p}.$$

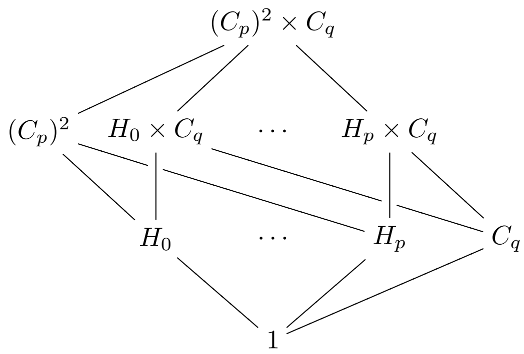

FIGURE 1. The subgroup lattice of $(C_p)^2 \times C_q$

It is easily seen that

$$\dim V = 2(p+2), \quad \dim W = 2(p+1),$$
$$\dim V^{H_i} = 4, \quad \dim W^{H_i} = 2,$$
$$\dim V^{H_i\times C_q} = 2, \quad \dim W^{H_i\times C_q} = 0,$$
$$\dim V^{C_q} = 2(p+1), \quad \dim W^{C_q} = 0.$$

Thus we see

$$d(1) = d(H_i) = d(H_i \times C_q) = 0 \text{ and } d(C_q) = 2p.$$

This implies that (V, W) is a $\mathcal{C}_G$-pair. On the other hand, since $d(G) = -2 < 0$, the Borsuk-Ulam inequality does not hold.

3.2. **The case of $G = C_p \times C_{p^2}$.** Let a and b be generators of C_p and C_{p^2} respectively. The nontrivial cyclic subgroups of G are the following:

- $H = \langle a \rangle$, isomorphic to C_p,
- $K_i = \langle a^i b^p \rangle$, $0 \le i \le p-1$, isomorphic to C_p,
- $L_i = \langle a^i b \rangle$, $0 \le i \le p-1$, isomorphic to C_{p^2}.

Setting $M = \langle a, b^p \rangle$, we define G-representations V and W to be

$$V = U_{L_0} \oplus U_{L_1} \oplus \cdots \oplus U_{L_{p-1}} \oplus (p+1)U_M,$$
$$W = 2U_H \oplus 2U_{K_1} \oplus \cdots \oplus 2U_{K_{p-1}}.$$

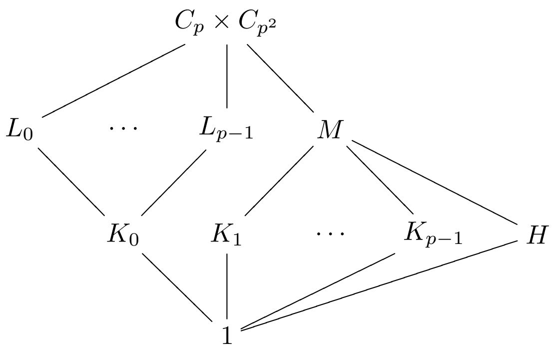

FIGURE 2. The subgroup lattice of $C_p \times C_{p^2}$

Noting obvious inclusions

$$K_0 \le L_i \le G \quad (0 \le i \le p-1),$$
$$K_i \le M \le G \quad (1 \le i \le p-1),$$
$$H \le M \le G,$$

we see

$$d(L_i) = 0 \quad (0 \le i \le p-1),$$
$$d(K_0) = 4p,$$
$$d(K_i) = 2(p-2) \quad (1 \le i \le p-1),$$
$$d(H) = 2(p-2).$$

Therefore (V, W) is a $\mathcal{C}_G$-pair; however, since $d(G) = -2 < 0$, the Borsuk-Ulam inequality does not hold. Thus the proof of Theorem 1.1 is completed.

4. **Nonabelian examples**

A similar question can be considered in the case of nonabelian finite groups. Unfortunately we do not completely solve it, but we can provide some examples.

Proposition 4.1. *Let G be a dihedral group D_n of order $2n$. If (V,W) is a $\mathcal{C}_G$-pair, then the Borsuk-Ulam inequality*

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds.

Proof. Recall the formula in Corollary 2.2:

$$d(G) = \sum_{1 \neq C \in \mathcal{C}_G} \frac{|C|}{|G|} m(C) d(C).$$

It suffices to show that $m(C) \geq 0$ for $C \in \mathcal{C}_G^0$. Set

$$D_n = \langle a, b \,|\, a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

The cyclic subgroups of D_n are as follows:

$$C_d = \langle a^{n/d} \rangle \text{ for } d \,|\, n, \quad E_k = \langle a^k b \rangle \cong C_2 \text{ for } 0 \leq k \leq n.$$

Fix any C_d $(1 \neq d \,|\, n)$. Since any cyclic subgroup including C_d is included in a unique maximal cyclic subgroup C_n, it follows that $m(C_d) = 0$ if $1 \neq d \,|\, n$ and $m(C_n) = 1$. Since a cyclic subgroup including E_k is only E_k itself, it follows that $m(E_k) = 1$. Therefore we have $d(G) \geq 0$. □

Proposition 4.2. *Let $G = \mathrm{PSL}(2, q)$, where q is a power of a prime p. If (V,W) is a $\mathcal{C}_G$-pair, then the Borsuk-Ulam inequality*

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds.

Proof. A similar argument in [5] shows that $m(C) \geq 0$ for any nontrivial cyclic subgroup C of G. See [5] for the details. □

Next consider a finite groups consisting of prime order elements and the unit element. Such groups are classified by [2]. We call them *groups of prime order elements*, and an elementary abelian group $(C_p)^k$, a metacyclic group $Z_{p,q}$, where p, q are primes and $q \,|\, p-1$, and the alternating groups A_4, A_5 are examples of groups of prime order elements.

Proposition 4.3. *Let G be a group of prime order elements. If (V,W) is a $\mathcal{C}_G$-pair, then the Borsuk-Ulam inequality holds.*

Proof. In this case, clearly $m(C) = 1$ for any nontrivial cyclic subgroup C, hence $d(G) \geq 0$. □

On the other hand, the quaternion group Q_8 is a nonabelian counterexample. Set $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ as a subset of quaternions $\mathbb{H}$. All proper subgroups are normal and cyclic: $D_1 = \langle i \rangle$, $D_2 = \langle j \rangle$, $D_3 = \langle k \rangle$, $C_2 = \langle -1 \rangle$ and 1.

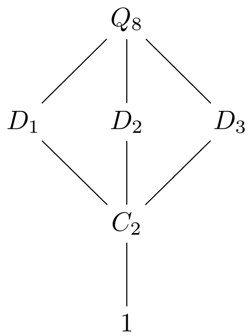

FIGURE 3. The subgroup lattice of Q_8

The quaternion group Q_8 has four 1-dimensional irreducible representations U_0, U_1, U_2, U_3, where U_0 denotes the trivial representation and $\operatorname{Ker} U_i = D_i$ for $i = 1, 2, 3$. There is a one 4-dimensional (orthogonal) representation $\mathbb{H}$ with the standard Q_8-action. Let $W = \mathbb{H}$ and $V = 2U_1 \oplus 2U_2 \oplus 2U_3$. Then $d(C_2) = 4$, $d(D_i) = 0$ for $i = 1, 2, 3$, whereas $d(G) = -2 < 0$.

Finally, we show Theorem 1.2.

Proof of Theorem 1.2. For a G-representation V and $C \in \mathcal{C}_G$, we denote by $V - V^C$ the complement of V^C in V as a C-representation and by $S(V - V^C)$ the unit sphere of $V - V^C$. Since $S(V) \smallsetminus S(V)^C$ is C-homotopy equivalent to $S(V - V^C)$, it turns out that there is a C-map $\tilde{f}_C : S(V - V^C) \to S(W - W^C)$. Since any $C \in \mathcal{C}_G^0$ is of prime order, C acts freely on $S(V - V^C)$ and $S(W - W^C)$. The Borsuk-Ulam theorem for C_p-maps (see [3], for example) asserts that

$$\dim S(V - V^C) \leq \dim S(W - W^C).$$

Therefore we obtain that (V, W) is a $\mathcal{C}_G$-pair and thus the Borsuk-Ulam inequality

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds by Proposition 4.3. □

As a finial remark, we notice the following fact. If there is a G-isovariant map

$$f : S(V) \to S(W),$$

then there is a C-map

$$f_C : S(V) \setminus S(V)^C \to S(W) \setminus S(W)^C$$

for every $C \in \mathcal{C}_G$; however, the converse is not correct; in fact, even a G-map $f : S(V) \to S(W)$ does not exist in general. For example, let $G = C_p \times C_p$. Then $\mathcal{C}_G^0$ consists of $p+1$ cyclic subgroups of order p, say $H_1, H_2, \ldots, H_N$ $(N = p+1 \geq 3)$. Set

$$V = 2U_{H_1} \oplus U_{H_2} \oplus U_{H_3}, \quad W = U_{H_1} \oplus 2U_{H_2} \oplus 2U_{H_3}.$$

Then H_i acts freely on $S(V - V^{H_i})$ and $S(W - W^{H_i})$, and

$$\dim S(V - V^{H_i}) \leq \dim S(W - W^{H_i}).$$

Hence one can easily construct an H_i-map

$$\bar{f}_{H_i} : S(V - V^{H_i}) \to S(W - W^{H_i})$$

for every $H_i \in \mathcal{C}_G^0$, $1 \leq i \leq N$. Extending $\bar{f}_{H_i}$, one has an H_i-map

$$f_{H_i} : S(V) \setminus S(V)^{H_i} \to S(W) \setminus S(W)^{H_i}.$$

On the other hand, there are no G-maps from $S(V)$ to $S(W)$. In fact, if there would be a G-map $f : S(V) \to S(W)$, then, by H_1-fixing, a C_p-map

$$f^{H_1} : S(2U_{H_1}) = S(V)^{H_1} \to S(W)^{H_1} = S(U_{H_1})$$

would be obtained; however, this contradicts the Borsuk-Ulam theorem which asserts that $\dim S(V)^{H_1} \leq \dim S(W)^{H_1}$. (See for example [4] for the Borsuk-Ulam theorem for $(C_p)^k$-maps.)

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